Exercise/Review for Eigen-problems

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1 Standard Hermitian Eigenvalue Problem

Let $A \in \mathbb{C}^{n\times n}$ *be Hermitian.*

A subspace $\mathcal{X} \in \mathbb{C}^n$ *is called an* invariant subspace *of* A *if* $A\mathcal{X} \subseteq \mathcal{X}$.

The Rayleigh quotient *of* A *with respect to a vector* $x \neq 0$ *is*

$$\rho(x) := \frac{x^{\mathrm{H}}Ax}{x^{\mathrm{H}}x}.$$

P-1.1. Show that

1. A is unitarily similar to a real diagonal matrix:

   $$A = U\Lambda U^{\mathrm{H}},$$

   where $U \in \mathbb{C}^{n\times n}$ is unitary and $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is real and diagonal. These λ_i are the eigenvalues arranged, for convenience, as

   $$\lambda_1 \le \lambda_2 \cdots \le \lambda_n.$$

   Write $U = [u_1, u_2, \ldots, u_n]$. Then u_i is the eigenvector corresponding to λ_i.

2. $\lambda_1 = \min_{x\neq 0} \rho(x), \quad \lambda_n = \max_{x\neq 0} \rho(x).$

Notations U and λ_i will be assigned as in this problem for the rest of this section.

P-1.2. 1. Show that $\mathcal{Y} \in \mathbb{C}^n$ is an invariant subspace of A if and only if there exists a square matrix M such that

   $$AY = YM,$$

   where the columns of Y consist of basis vectors of $\mathcal{Y}$. How are the eigenvalues and eigenvectors of M and those of A are related?

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2. Let $V = [u_1, u_2, \dots, u_k]$. Find the eigen-decomposition of $A + \xi VV^{\mathrm{H}}$. What are its eigenvalues and eigenvectors?

P-1.3. Let $X \in \mathbb{C}^{n\times k}$ satisfying $X^{\mathrm{H}}X = I_k$. Define $\mathscr{R}(H) = AX - XH$. Show that

$$\|\mathscr{R}(X^{\mathrm{H}}AX)\|_{\mathrm{F}} \le \|\mathscr{R}(M)\|_{\mathrm{F}}$$

for any Hermitian $M \in \mathbb{C}^{k\times k}$, with equality if and only if $M = X^{\mathrm{H}}AX$, where $\|\cdot\|_{\mathrm{F}}$ stands for the matrix Frobenius norm.

2 Generalized Hermitian Eigenvalue Problem

Let $A, B \in \mathbb{C}^{n\times n}$ be Hermitian with $B \succ 0$ (positive definite). We are interested in the generalized eigenvalue problem for the matrix pencil $A - \lambda B$.
The Rayleigh quotient *of $A - \lambda B$ with respect to a vector $x \neq 0$ is*

$$\rho(x) := \frac{x^{\mathrm{H}}Ax}{x^{\mathrm{H}}Bx}.$$

In an optimization approach, it is often needed to solve

$$\inf_t \rho(x + tp),$$

the so-called Line Search*, where t is either among $\mathbb{R}$ or $\mathbb{C}$.*
A subspace $\mathcal{X} \in \mathbb{C}^n$ is called an invariant subspace *of $A - \lambda B$ if $A\mathcal{X} \subseteq B\mathcal{X}$.*

P-2.1. Show that

1. There exists nonsingular $U \equiv [u_1, u_2, \dots, u_n] \in \mathbb{C}^{n\times n}$ such that

$$U^{\mathrm{H}}AU = \Lambda, \quad U^{\mathrm{H}}BU = I_n,$$

where $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ is real and diagonal. These λ_i are the eigenvalues of $A - \lambda B$ arranged, for convenience, as

$$\lambda_1 \le \lambda_2 \cdots \le \lambda_n$$

with the corresponding eigenvectors u_i.

2. $\lambda_1 = \min_{x\neq 0} \rho(x), \quad \lambda_n = \max_{x\neq 0} \rho(x)$.

Notations U and λ_i will be assigned as in this problem for the rest of this section.

P-2.2. 1. Show that $\mathcal{Y} \in \mathbb{C}^n$ is an invariant subspace of $A - \lambda B$ if and only if there exists a square matrix M such that

$$AY = BYM,$$

where the columns of Y consist of basis vectors of $\mathcal{Y}$. How are the eigenvalues and eigenvectors of M and those of $A - \lambda B$ are related?

2. Let $V = [u_1, u_2, \dots, u_k]$. Find the eigen-decomposition of matrix pencil

$$A + \xi(BV)(BV)^{\mathrm{H}} - \lambda B.$$

What are its eigenvalues and eigenvectors?

P-2.3. Let $X \in \mathbb{C}^{n\times k}$ satisfying $X^{\mathrm{H}}BX = I_k$. Define $\mathscr{R}(H) = AX - BXH$. Show that

$$\|B^{-1/2}\mathscr{R}(X^{\mathrm{H}}AX)\|_{\mathrm{F}} \le \|B^{-1/2}\mathscr{R}(M)\|_{\mathrm{F}}$$

for any Hermitian $M \in \mathbb{C}^{k\times k}$, with equality if and only if $M = X^{\mathrm{H}}AX$.

P-2.4. Let $x, p \in \mathbb{C}^n$ are nonzero vectors.

1. Verify that the gradient of ρ at a point x is

$$\nabla\rho(x) = \frac{2}{x^{\mathrm{H}}Bx}\,[A - \rho(x)B]x \equiv \frac{2}{x^{\mathrm{H}}Bx}\,r(x).$$

(Be mindful about the distinction between the real case and complex case.)

2. In the complex case, it is possible that

$$\inf_{t\in\mathbb{R}} \rho(x + tp) > \inf_{t\in\mathbb{C}} \rho(x + tp).$$

Find an example.

3. Verify that

$$\inf_{t\in\mathbb{C}} \rho(x + tp) = \min_{|\xi|^2+|\zeta|^2>0} \rho(\xi x + \zeta p)$$

which is the smaller eigenvalue of $[x, p]^{\mathrm{H}}(A - \lambda B)[x, p]$, provided x and p are linearly independent.

4. Suppose $p^{\mathrm{H}}Bx = 0$ and $p^{\mathrm{H}}Ax \ne 0$. Show that

$$\inf_{t\in\mathbb{C}} \rho(x + tp) = \min_{t\in\mathbb{C}} \rho(x + tp) < \min\{\rho(x), \rho(p)\}.$$

3 Programming Assignment

These assignments will be posted at

`www.uta.edu/faculty/rcli/G2S3/g2s3.html`.