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50 Years of ADI Methods: Celebrating the Contributions of Jim Douglas, Don Peaceman, and Henry Rachford

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Physical intuition often provides the motivating spark for the development of mathematical methods that, on generalization to an abstract form, are used far beyond their initially intended purpose. This is the case for the alternating direction implicit (ADI) scheme, first proposed for the implicit solution of heat flow (a parabolic partial differential equation) in two geometric dimensions. Donald Peaceman and Henry Rachford (1955), and Jim Douglas Jr. and Rachford (1955), were able to fit the problem into the limited computers available at the time by splitting the time-step procedure into two fractional steps. They solved first for the flow in one direction, while holding the orthogonal flow fixed, and then for flow in the other direction, while holding the original flow fixed. Interestingly, each individual operator produced a simple tridiagonal matrix. It is perhaps a little surprising that such a simple method would work, but it did. The method was quickly extended to three dimensions by Douglas and Rachford (1956), Brian (1961), and Douglas (1962), and Douglas, Peaceman, and Rachford proved both stability and convergence for the methods.

More importantly, though, the use of ADI to solve linear matrix equations led to the first viable multidimensional petroleum reservoir simulators. Reservoir simulation is among the multiscale systems of many types that require implicit discretization. To solve such a problem of any useful size, memory-efficient, fast converging methods are needed to solve the large linear equations that arise at each time-step.

Rice University, November 2005. *Fifty years after their pioneering work on alternating direction implicit methods, Jim Douglas Jr., Donald Peaceman, and Henry Rachford attended a conference organized to honor them and celebrate a legacy that continues to grow.*

The evolution of techniques for solving linear equations that arise in reservoir simulation has continued, with newer techniques displacing older ones—from Stone's strongly implicit procedure (SIP) to line successive over-relaxation (LSOR) to the Newton–Krylov schemes with ILU-type preconditioners in use today and the multiscale, multigrid solvers now starting to come of age. By the late 1960s ADI was no longer used in reservoir simulation. It was not effective in the multiphase models or the models with highly heterogeneous properties common in real reservoirs.

In other ways, however, ADI and its offspring are very much alive and kicking. The ADI method itself is used heavily in a great variety of applications—from astrophysical and bioengineering applications to tsunami modeling and Black–Scholes option pricing. It is memory-efficient and easy to parallelize, and it has adapted well to evolving software and hardware architecture.

Furthermore, the generalization of operator-splitting methods took on a life of its own. It was immediately understood that operator splitting for many initial value problems (IVPs) could be considered in the generalized form $d\varphi/dt + A(\varphi) = 0$ with the possibly multivalued operator $A = \sum_{j=1}^{J} A_i$. For $J = 2$, the Peaceman–Rachford scheme is of the backward-Euler type for A_1 and of the forward-Euler type for A_2 on the time interval $[t^n, t^{n+½}]$, with the situation reversed on $[t^{n+½}, t^{n+1}]$, and is similar to the Crank–Nicholson method. The Douglas–Rachford scheme, which is of a predictor–corrector type, can easily be generalized to more than two operators ($J > 2$). These schemes, which have been around for 50 years, have motivated a large body of literature, either as methods for approximating the solution of time-dependent problems or as iterative methods for solving linear and nonlinear steady-state problems in finite or infinite dimension.

Fifty years after the publication of the original paper, Rice University, the University of Texas at Austin, the University of Houston, and ExxonMobil's Upstream Research Company (the descendant of the Humble Oil production research facility) organized a conference to honor the achievements of the developers of ADI and to recognize the importance of operator-splitting (OS) methods. The conference, initiated in large part by Richard Tapia of the Rice CAAM department, was held on the campus of Rice University in November 2005 (http://ceee.rice.edu/meetings/dpr/index.html). Just over a hundred attendees from academia and industry participated.

Two days of talks about work both past and present revealed the extent to which ADI and OS methods have permeated the field of computational mathematics. Presenting plenary talks on the context of ADI in today's world were R. Glowinksi (University of Houston) from an academic perspective and J.W. Watts (ExxonMobil) from an industrial perspective. Glowinksi traced the evolution of OS methods, which proliferated like branches from a tree; see, for example, Glowinski's article in the *Handbook of Numerical Analysis*, Vol. IX. Many speakers discussed modern extensions of alternating-direction methods based on finite element methods, as well as the use of ADI as a smoother for multigrid methods. From the fractional step θ-scheme, introduced in the 1980s, to the family of OS schemes, including Lie's, Strang's, and Marchuk–Yanenko's, ADI's relationship to the world of OS was emphasized. Lie's basic extension assumes that the IVP operator is linear, leading to a solution with an exponential form, $\varphi(t+\tau) = e^{-A\tau}\varphi(t)$. It is first-order accurate and unconditionally stable if the A_j operators are monotone. The Strang scheme is second-order accurate. The Marchuk–Yanenko scheme implements Lie's scheme, using just one step of the backward Euler scheme to discretize each subproblem, and is order-τ accurate. Glo-winski further demonstrated that both the Peaceman–Rachford and the Douglas–Rachford schemes could be derived from an augmented Lagrangian approach.

Given a complex, multiphysics, multiscale computational model, it is sometimes, but not always, apparent how an operator can be split. For the original ADI method, the split was geometric. For some problems, it is possible to split the calculation so that physical models are solved independently. In such cases, time splitting can also make it possible to couple independent codes, allowing simultaneous use of disparate, best-in-class simulation codes.

The development of ADI also epitomizes the best sort of academic and industrial collaboration—something pioneered in the U.S. and widely emulated around the globe. In pursuit of solutions to scientific problems arising from industrial applications, researchers with mutual interests and complementary skills and resources push the boundaries of applied math. The mid-1950s was a time of much early work on computational methods for solving partial differential equations. The energy industry's need for computationally efficient numerical algorithms presaged those of many industries that followed. Today, industries from energy and biotech to IT and economics rely heavily on advances provided by computational analysis, and the legacy of Douglas, Peaceman, and Rachford continues to grow.

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