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An Enjoyable Excursion to the Frontiers of Chaos

From Calculus to Chaos: An Introduction to Dynamics. *By David Acheson, Oxford University Press, Oxford, New York, and Melbourne, 1997, ix + 269 pages, $27.95.*

The author states, in his admirably Reaganesque (single-page) preface, that

> "My main aim is to help people see, and actually *enjoy*, some truly remarkable applications of mathematics, and the best way I know of doing that is to introduce, by means of simple examples, some of the most exciting results and discoveries, in such a way that the really big ideas do not get obscured in a snowstorm of detail. To this end, we move along quite rapidly, from first steps to frontiers."

That sounded like a pretty tall order to this reviewer, whose carefully cultivated sales-resistance was no match for the promise of "really big ideas" and "remarkable applications," as well as "exciting results and discoveries" leading all the way to "frontiers" of a subject with which every victim of technical training has at least a nodding acquaintance.

BOOK REVIEW
By James Case

Had I read no farther, I would already have been reminded of the considerable difficulties the ancient Greeks had with the concept of continuous motion—difficulties immortalized in Zeno's various paradoxes—and of the still unresolved issue of gravity's alleged ability to cause instantaneous action across arbitrarily large distances. Acheson, rather wisely, circumvents these age-old difficulties by picking up the story in August of 1684, with Edmund Halley's fruitful journey to Cambridge to learn Newton's opinion concerning the motion of a single planet orbiting the sun under the influence of a force proportional to the inverse square of the separating distance. Halley was reportedly delighted to learn that, according to Newton's already completed calculations, the orbit would indeed be an ellipse. Newton was also able to tell him, during that visit, that Kepler's second law (whereby a line drawn between the two bodies sweeps out equal areas in equal times) holds under any "central force."

Acheson goes on to point out that, although Newton studied ordinary differential equations and their power-series solutions in some detail, it was Euler who first wrote the differential equations of particle motion and used them in a systematic way. He even produces the paragraph containing those equations from a paper by Euler in the memoirs of the Berlin Academy for 1749. The equations assert, among other things, that the force acting on an individual particle, along with its acceleration, can be resolved into components parallel to a fixed set of coordinate axes. Although this technique is commonplace today, and was known to both Newton and Galileo, most of Euler's contemporaries apparently learned it from him.

Exercises are included at the end of every chapter, and answers are provided at the back, presumably to make the book suitable for classroom use. It is less than clear, however, where such a course might fit in any standard curriculum. It could certainly be used for independent study—most profitably by those with access to personal computers—or in a summer "enrichment" course for high school teachers. It could even be used by those with no previous computing experience, as it includes what the author describes as "an unusually down-to-earth introduction to the whole matter. This renders the book accessible, in the author's and this reviewer's opinion, to any "general reader" who knows a little calculus and "refuses to be put off by a few equations."

Chapter 1, the introduction, is little more than a catalogue of topics to be treated later. However, solving the problem of a projectile in a vacuum, where the equations of motion are trivial, the author does manage to illustrate the manner in which problems of dynamics are ordinarily analyzed. The chapter also reproduces a page from Galileo's working papers on motion, containing actual measurements of the parabolic arcs traversed by (heavy) balls released from a common (elevated) position with various horizontal velocities.

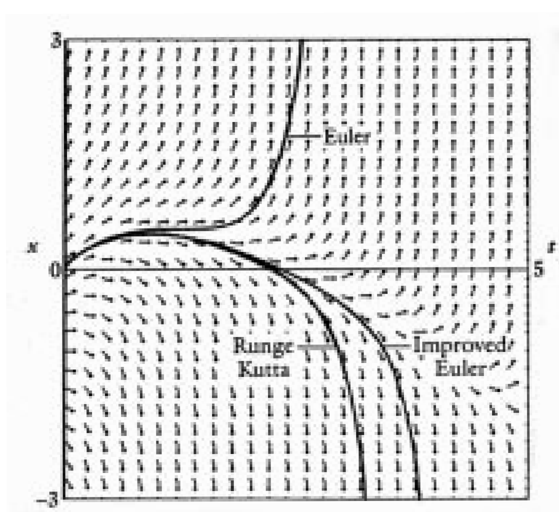

Figure 1. *One of the many vector-field diagrams that help make the book under review a self-contained introduction to differential equations.*

Chapter 2 presents a short review of elementary calculus. One suspects that it was written after the rest of the book was completed, as it contains almost nothing more than is strictly necessary for subsequent developments. The emphasis is on Taylor series, special forms of which were known to—and used effectively by—Newton and others long before Brook Taylor published the general result in 1715. Acheson fixes priority by reproducing the title page of Newton's 1671 treatise *The Method of Fluxions and Infinite Series,* which remained unpublished and untranslated (from the original Latin) until 1736. He also includes sketches from Newton's unpublished manuscript of 1684, *De Motu Corporum in Gyrum*, in which he made essential use of the calculus (as compared with his magnum opus, the *Principia*, which relied mainly on geometric arguments).

Acheson assumes no prior knowledge of differential equations—Chapters 3 and 4 form a self-contained introduction to the subject, with the focus on analytic solution methods in the former and on numerical methods in the latter. Both treatments are brief and to the point, making extensive use of vector-field diagrams of the sort found in Figure 1. Such diagrams can now be obtained at a keystroke from most symbolic manipulators, such as Derive.

The field shown in Figure 1 (from the book, page 52) corresponds to the ODE $\dot{x} = (1 + t)x + 1 - 3t + t^2$, discussed by Newton himself in his treatise on fluxions. He obtained the power-series solution $x = t - t^2 + t^3/3 - t^4/6 + \ldots$ for the case $x(0) = 0$. The solution curves in the figure correspond to the slightly different case $x(0) = 0.0655$, and were obtained with a step-size $h = 0.035$. Acheson describes the Euler and "improved Euler" (trapezoidal) methods at some length, before presenting (without derivation or even much explanation) the standard Runge–Kutta method.

An appendix containing a number of Visual Basic routines—oriented toward specific problems—offers all three methods as user options, with which the reader is invited to experiment. Although my computer does not offer Visual Basic, I did attempt a few of the simpler exercises using Derive, and obtained results similar to those reported by Acheson.

The first four chapters are preliminary; the book really begins with Chapter 5: "Elementary Oscillations." Acheson's discussion begins, as seemingly it must, with small, approximately linear oscillations of the sort executed by the pendulum in a grandfather clock. After the usual discussion of simple and linked oscillators, multiple modes of oscillation, and the small vibrations of a double pendulum, however, the discussion moves along rather quickly to the large nonlinear oscillations such a pendulum might perform if unconstrained by its casing. These motions, which are ordinarily ignored in introductory treatments of the subject, are investigated with the aid of numerical methods in the cylindrical phase space natural to the problem, without recourse to the (analytic) first integral. What would once have seemed a time-consuming digression becomes, with the aid of numerical methods, readily accessible.

Chapter 6, on planetary motion, begins with Kepler's laws, particularly the third, which asserts that the periods of various orbits are in proportion to the 3/2 power of their mean distances from the sun. The discovery of this law was historically important in that it furnished an additional empirically testable hypothesis and helped point the way to the inverse square law of gravitational attraction. In addition to the familiar analytic derivation of the "equal areas" rule, Acheson includes a page from Newton's *De Motu* containing a "discrete impulse" derivation of the rule in which an analysis in terms of Euclidean geometry is followed by an implicit passage to the limit.

The chapter contains the first of Acheson's promised "frontiers," in the form of an extended discussion of the three-body problem. This is introduced as a natural extension of the two-body problem, which enters the discussion via the observation that the sun is not really an immobile center about which the planets may orbit. It moves too, if only slightly, under the gravitational forces exerted by the planets. Perhaps the chapter's most vivid lesson concerns the near collisions that can be experienced by three mutually gravitating point masses, and the kinds of orbits likely to result.

In particular, Acheson offers an example in which bodies 1, 2, and 3 lie initially in a plane, with velocities tangent to that plane. After the first near collision, masses 1 and 3 travel together as a pair, spinning rapidly about one another, while 2 wanders off on its own for a while before returning to steal 3 away from 1. Thereafter, 2 and 3 travel as a pair, again spinning rapidly about one another, until 1 returns to steal 2 away from 3. Then 1 and 2 travel as a pair for a time, while 3 wanders off on its own. And so on. All this is the output of a program called THREEBP, consisting of only a few dozen lines of code!

After a chapter on waves and diffusion, and another on variational principles and the Lagrange equations, comes Chapter 9: "Fluid Flow." This is a minor masterpiece of exposition, proceeding from ideal flows through Reynolds numbers and viscosity, to a discussion of singular perturbations and boundary layer phenomena, featuring the separation diagram from Prandtl's original (1905) paper on the subject. Along the way it describes the spin-down of a stirred cup of tea, swimming spermatozoa, and the reversibility of the solutions of the Navier–Stokes equations relating to highly viscous flows—all in terms an attentive beginner would find readily comprehensible.

The following chapter, on the instability of motion, begins with a description of Osborne Reynolds's laminar flow experiment of 1883, revealing the suddenness with which turbulence can set in. It then proceeds through a discussion of linear stability and Euler buckling to simple bifurcation diagrams, structural stability, and sudden (catastrophic) changes of state, before returning to the original Reynolds experiment. Acheson points out that recent improvements to apparatus and technique show how, if truly extra-ordinary pains are taken to eliminate even the most minor destabilizing shocks, Reynolds flow can be kept stable for values of the Reynolds number as high as 10^5. Once turbulent motion has begun, however, that critical number has to be cut all the way back to 2000, or thereabouts, before laminar flow resumes. In other words, the system exhibits a violent—and as yet unexplained—form of hysteresis.

The most distant of the frontiers visited in the book are found in the last two chapters. The chapter titled "Nonlinear Oscillations and Chaos" begins with a discussion of forced nonlinear oscillators, the van der Pol equation, and the conditions required for chaos. Next come the Lorenz equations, followed by period-doubling and stretch-and-fold phenomena as they occur in the Rössler equations. In this way, Acheson contrives to summon a remarkably brief—yet surprisingly comprehensive and comprehensible—introduction to the emerging science of chaos from a background of familiar techniques and phenomena.

Acheson opens the final chapter, "The Not So Simple Pendulum," with the intriguing fact (discovered in 1908) that an inverted pendulum can be stabilized in the upright position if the pivot is made to oscillate vertically. Indeed, if the pivot elevation is $\zeta(t) = a \cos \omega t$, the stability condition is $\omega > \sqrt{2g\lambda} / a$, where g is the acceleration of gravity and λ is the length of the pendulum. Long intrigued by this surprising result, Acheson revisited the problem a few years ago, to discover that the same "trick" can be accom-plished with any finite number of linked pendulums, balanced one on top of another! In fact, he demonstrated that there exists a sizeable triangular region (with acurvi-linear hypotenuse) in the $a\omega$-plane that restabilizes such a system, even after severe disturbances. Of late, his conclusions have been confirmed experimentally, with seemingly remarkable accuracy!

I enjoyed reading this book and learned quite a lot from it. I recommend it to anyone who—like myself—knows calculus better than chaos, and would like to begin rectifying the situation as painlessly as possible.

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