from *SIAM News*, Volume 32, Number 6

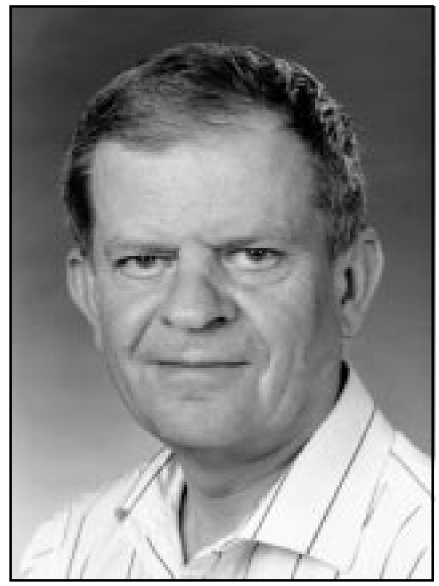
Askey, Rokhlin Elected to NAS

Richard Askey, John Bascom Professor of Mathematics at the University of Wisconsin, Madison, and Vladimir Rokhlin, a professor of mathematics and computer science at Yale University, were among the 60 new members elected to the National Academy of Sciences in April. In addition to the two SIAM members, the mathematical sciences community was also represented by Elwyn Berlecamp and Vaughan Jones, both of the University of California, Berkeley; Dusa McDuff, of the State University of New York, Stony Brook; and, as a foreign associate, Yakov Sinai, a professor of mathematics at Princeton University, from Russia.

Leading Expert in Special Functions

Anyone studying or using special functions can expect to encounter the name Richard Askey. A leading expert in the field, Askey is a long-time member of the SIAM Activity Group on Orthogonal Polynomials and Special Functions. As a member of the editorial board of *SIAM Journal on Mathematical Analysis* from its first issue (1970) until 1993, he attracted some important papers on special functions to the journal.

The Askey table of "classical" orthogonal polynomials is a very clever collection and arrangement of various hypergeometric orthogonal polynomials that have additional, and useful, structure. At the bottom of the table are the very classical orthogonal polynomials known as Hermite polynomials, Laguerre polynomials, and Jacobi polynomials; apart from their orthogonality, these polynomials have various useful properties involving the differential operator (such as raising and lowering operators, linear second-order differential equation of the Sturm–Liouville type, Rodrigues formula).



Beyond his considerable research accomplishments, Richard Askey, according to his colleagues, is the perfect person to consult if you are interested in using special functions to solve a particular problem.

Replacing the differential operator by difference operators and q-difference operators on linear, quadratic, and exponential lattices gives various important orthogonal polynomials, such as the Krawtchouk polynomials (orthogonal with respect to the binomial distribution), Meixner polynomials (orthogonal with respect to the Pascal distribution or negative binomial), Charlier polynomials (orthogonal with respect to the Poisson distribution), four sets of Hahn polynomials, and two sets of polynomials named after James Wilson (a former PhD student of Askey's) and Giulio Racah (because of their relation to Wigner's 6-j symbols, for which the finite cases are known as Racah coefficients). All these polynomials have q-extensions, and Askey introduced a very important family of orthogonal polynomials from which all the others can be obtained by taking appropriate limits. These polynomials—the Askey–Wilson polynomials—are found at the very top of the Askey table. A detailed description of the Askey table can be found in a report by R. Koekoek and R.F. Swart-touw (ftp://ftp.twi.tudelft.nl/TWI/publications/tech-reports/1998/DUT-TWI-98-17.ps.gz).

Askey certainly is one of the main forces behind the surge of interest in q-extensions during the last few decades. Most special functions, and in particular all hypergeometric functions, have q-extensions, usually obtained by replacing Pochhammer coefficients $(a)_n = a(a+1)\cdots(a+n-1)$ by $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$, where q is usually real and $|q| < 1$ (but this is certainly not always the case).

One very important reason for studying q-series is that they can be used to do refined counting. Partitions, i.e., counting the number of ways a positive integer can be written as a sum of integers, are just one of the ways in which q-series can be used in counting. Another important application is counting with noncommuting variables: If a and b are noncommuting, but $ab = qba$, then q-series can be used to expand the binomium $(a + b)^n$, extending Newton's binomium. Such noncommuting variables are very prominent in mathematical physics (the minor industry that has developed around q-extensions of the harmonic oscillator uses various q-extensions of the Hermite polynomials) and quantum group constructions. Askey evaluated several beta-integrals and q-extensions of these beta-integrals and explained the natural setting in which they arise. The orthogonality of the Askey–Wilson polynomials, for example, follows from a clever evaluation of a q-beta integral.

Such q-extensions are not new: At the end of the 19th century, Leonard James Rogers wrote a series of papers on q-extensions of ultraspherical polynomials. Askey, with George Andrews and Mourad Ismail, deciphered Rogers's papers and realized that Rogers had obtained a lot of remarkable results. One of them, on connection coefficients for these q-ultraspherical polynomials, surprisingly gave a set of identities that had been conjectured by Ramanujan and are now known as the Rogers–Ramanujan identities. The q-ultraspherical polynomials are the q-analogs of the ultraspher-ical polynomials (Gegenbauer polynomials), but Rogers did not know that they were orthogonal polynomials. Askey and Ismail thoroughly studied the q-ultraspherical polynomials and found the orthogonality relations. Bill Allaway, Waleed Al-Salam, and Askey later observed that by taking q a root of unity, it is possible to get interesting new families of orthogonal polynomials: the sieved ultraspherical polynomials.

Askey's name is also connected to de Branges' proof of the Bieberbach conjecture, in which some crucial inequalities turned out to be special cases of inequalities for sums of certain Jacobi polynomials that had been obtained a few years earlier by Askey and George Gasper; Askey and Gasper were proving the positivity of Cesáro $(C, \alpha + \beta + 2)$ means of the Jacobi series of a nonnegative function.

Askey has been a matchmaker in a number of cases. If you are interested in using special functions to solve a particular problem, Askey is the perfect person to put you in touch with someone who can solve the problem at hand. One example is a Berkeley astronomer who called Askey, wanting to know if there are orthogonal polynomials in two variables that are orthogonal on a regular hexagon. Askey pointed out that two people—Tom Koornwinder and Charles Dunkl—had done deep work on polynomials that are orthogonal on regions with some symmetry. The astronomer contacted Dunkl, who found a set of recurrence relations to generate these polynomials. The polynomials were used in the fabrication of mirrors for the telescope at the W.M. Keck Observatory (Mauna Kea, Hawaii), which consists of hexagonal optical elements. (Dunkl's paper on that work appeared in *SIAM Journal on Applied Mathematics*, 47 (1987), pages 343–351.)

Askey has always been a very useful source of information on special functions. Many consider his lecture notes, *Orthogonal Polynomials and Special Functions* (SIAM Regional Conference Series in Applied Mathematics, Vol. 21, 1975), to be one of the reasons for the renewed interest in research in orthogonal polynomials at the end of the seventies. Orthogonal polynomials were one of the favorite research areas of Gabor Szegö, whose book *Orthogonal Polynomials* is still a very useful resource. Askey did a terrific job in editing Szegö's collected papers (Birkhäuser, Boston, 1982), in which each paper is accompanied by useful background information. Askey's interest in the history of mathematics is also reflected in *A Century of Mathematics in America* (edited by Peter Duren, with assistance from Richard Askey and Uta Merzbach; American Mathematical Society, 1991); the third volume contains Askey's contribution, "Handbooks of Special Functions."

Askey is closely involved in two current projects for handbooks in special functions: the Digital Library of Mathematical Functions (http://math.nist.gov/DigitalMathLib) and an update of the Bateman Manuscript (the so-called Askey–Bateman project). (See Barry Cipra's article in the March 1998 issue of *SIAM News* for additional information about both projects.) Askey has also done seminal work in describing Ramanujan's work on special functions to the mathematical community. In recent years, Askey has been very active in matters of mathematical education.

Askey's most recent book, with George Andrews and Ranjan Roy, *Special Functions* (*Encyclopedia of Mathematics and its Applications*, Vol. 71, Cambridge University Press, 1999), should be on the desk of anyone interested in special functions. An article on the mathematical contributions of Richard Askey, written by Gasper, Ismail, Koornwinder, Paul Nevai, and Dennis Stanton, will appear in *q-Series from a Contemporary Perspective* (Contemporary Mathematics, American Mathematical Society).

■ ■ ■

At the Forefront of New Developments in Applied Mathematics

Vladimir Rokhlin has had a profound impact on scientific computing and applied mathematics, most notably through his work in developing "analysis-based fast algorithms." These algorithms include the fast multipole method for the Laplace equation, the fast multipole method for the Helmholtz equation, and the nonequispaced fast Fourier transform. He has also made fundamental contributions to inverse scattering and approximation theory.

With his "analysis-based fast algorithms," including fast multipole methods, Vladimir Rokhlin has been at the forefront of an important development in applied mathematics. Photograph by Michael Marsland, Yale University, Office of Public Affairs.

Before turning to his specific technical contributions, it is worth observing that Rokhlin was the first person to take a systematic approach in combining approximation theory, the classical theory of special functions, and modern computer science to reduce the computational cost associated with the basic integral operators of mathematical physics. Earlier fast algorithms, like the fast Fourier transform, had (and continue to have) great impact, but they are brittle—they require uniform data structures and are unsuitable for complex geometry. An interesting consequence of the approximate nature of the new class of methods is that they are more flexible and more robust than their predecessors.

Develoment of the fast multipole method (FMM) for the Laplace equation was the work of slightly more than a decade, beginning with [5] and nearing completion with [3]. For high-frequency scattering, the initial description was in [6], with extension to three dimensions in [7]. In both cases, the FMM relies on translation operators for multipole expansions, together with a hierarchical subdivision of space to compute "N-body interactions" in $O(N)$ or $O(N \log N)$ time, rather than $O(N^2)$. The analysis of these operators is particularly subtle for high-frequency scattering, where the fundamental tool is a new diagonal form for the translation of multipole expansions. Without this mathematical insight, there would be no fast algorithm.

The nonequispaced fast Fourier transform generalizes the classical fast Fourier transform (FFT) to the case of noninteger frequencies and nonequi-spaced nodes. This greatly enhances the applicability of the FFT. In particular, it allows for the efficient use of the continuous Fourier transform, when discretization at equispaced nodes is inappropriate. The consequences of this work are likely to be far-reaching.

In inverse scattering, Rokhlin and Y. Chen derived a new approach to computation, based on a stable trace formula (the first one of its kind) [1]. Finally, Rokhlin (with J. Ma and S. Wandzura) [4] has constructed a remarkable generalization of Gaussian quadrature for functions with a wide range of allowed singularities.

In summary, Rokhlin initiated and has been at the forefront of an important development in applied mathematics—one whose effect will be felt much more strongly in the years to come. Fast multipole methods, for example, are already becoming standard tools in the elec-tromagnetics, computational chemistry, and biophysics communities.

References

[1] Y. Chen and V. Rokhlin, *On the inverse scattering problem for the Helmholtz equation in one dimension*, Inverse Problems 8, 1992, 365–391.

[2] A. Dutt and V. Rokhlin, *Fast Fourier transforms for nonequispaced data,* SIAM J. Sci. Comput. 14, 1993, 1368–1393.

[3] L. Greengard and V. Rokhlin, *A new version of the fast multipole method for the Laplace equation in three dimensions*, Acta Num. 6, 1997, 229–269.

[4] J. Ma, V. Rokhlin, and S. Wandzura, *Generalized Gaussian quadrature rules for systems of arbitrary functions,* SIAM J. Numer. Anal. 33, 1996, 971–996.

[5] V. Rokhlin, *Rapid solution of integral equations of classical potential theory,* J. Comput. Phys. 60, 1985, 187–207.

[6] V. Rokhlin, *Rapid solution of integral equations of scattering theory in two dimensions*, J. Comput. Phys. 86, 1990, 414–439.

[7] V. Rokhlin, *Diagonal forms of translation operators for the Helmholtz equation in three dimensions*, Appl. Comput. Harmon. Anal. 1, 1993, 82–93.

Walter van Assche, a professor of mathematics at Katholieke Universiteit Leuven and vice chair of the SIAM Activity Group on Special Functions and Orthogonal Polynomials, provided the material on Richard Askey. Leslie Greengard of the Courant Institute of Mathematical Sciences contributed the section on Vladimir Rokhlin.