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In the Fold: Origami Meets Mathematics

By Barry A. Cipra

Say "origami" and most people picture birds, fish, or frogs made of sharply creased paper, an amusing exercise for children. But for participants in the Third International Meeting of Origami, Science, Math, and Education, held March 9–11 in Pacific Grove, California, the ancient Japanese word conjures up images of fantastically complex organic and geometric shapes, of axioms and algorithms, and of applications ranging from safer airbags in cars to the deployment in outer space of telescopes 100 meters or more in diameter.

"There's a particular appeal of origami to mathematicians," says Tom Hull, a mathematician at Merrimack College in Andover, Massachusetts, and the organizer of the conference. Paperfolding offers innumerable challenges, both theoretical and practical—and does so in a way that appeals to mathematicians' aesthetic sensibilities. "I tend to view origami as being latent mathematics," Hull says.

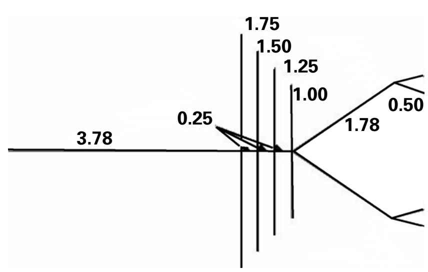
Unlimited Possibilities

Traditional origami starts with a square sheet of paper, typically white on one side and colored on the other. The origamist makes a sequence of "mountain" and "valley" folds, creating a network of creases that turn the square into a mosaic of polygonal facets. And as computer graphics enthusiasts know, you can do a lot with polygons.

Erik Demaine, a computer scientist at the University of Waterloo, described some of the amazing polygonal and polyhedral possibilities. A square of origami paper that's black on one side and white on the other, for example, can be folded to create a checkerboard pattern. Providing a much more general view is a theorem proved in 1999 by Demaine, his father, Martin, also a computer scientist at the University of Waterloo, and Joseph Mitchell of the State University of New York at Stony Brook: Given an arbitrarily complicated polyhedron with some faces black and the others white (or whatever two colors you like), there's a way to reproduce it from a single sheet of bicolored paper. The implication is that it's possible, at least in principle, to fashion a convincing origami zebra from a single square of paper.

"In principle" and "in practice" are often two different things, of course, but Robert Lang has brought them together. Lang, an engineer at JDS Uniphase, in Santa Clara, California, is also one of the leading origami designers in the world. He doesn't use much origami in his day job, but he does use engineering principles in designing origami. In particular, he has developed a computer algorithm, called TreeMaker, for the design of complicated origami objects, such as multilegged insects or an antlered moose (see Figure 1).

The input for TreeMaker is a two-dimensional stick figure that captures the essential features of the target object. Graph-theoretically, the stick figure is a weighted tree, the weights being the lengths of the various edges. The computer does the hard part: It works out a way to fold a square of paper so that the result, which Lang calls the "base," sits exactly over the stick figure. In the final (nonalgorithmic) step, standard origami techniques are used to flesh out the base, making the model aesthetically satisfying and realistic. Lang has used his algorithm to create a creepy–crawly zoo of beetles, spiders, and scorpions.



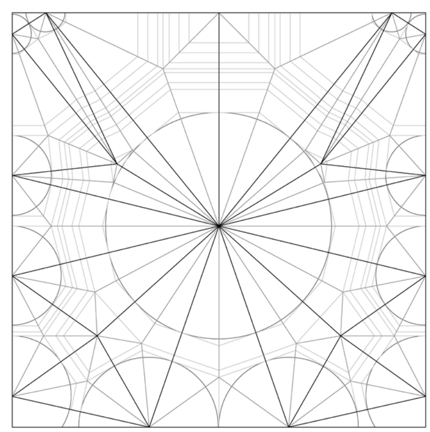


Figure 1. *TreeMaker sequence. Starting from a user-provided stick figure (above left), Robert Lang's TreeMaker algorithm produces an "optimal" crease pattern (above) for folding into a complex model (left)—in this case a scorpion.*

Actually, a base is easy to fold if you don't mind being wasteful: Just stand a large sheet of paper upright over the stick figure and make a fold at each vertex, pleating the sheet so that its shadow traces out the figure, with two plies per line segment. TreeMaker's approach, as you might expect, is much more clever.

"What TreeMaker does is come up with something that's anywhere from 5 to 50 times more efficient in its use of paper," Lang explains. It aims to make optimal use of the paper—i.e., to create a base for a given stick figure with the smallest possible square of paper, or, conversely, to make the largest possible (scaled up) base from a given square. It does so by formulating the design challenge as a problem in nonlinear constrained optimization, turning the weighted tree into a set of algebraic equations, and then using an off-the-shelf computer code called CFSQP, developed at the University of Maryland, to solve the equations.

"It finds a local maximum," Lang says. "And with a little bit of human intelligence, you can convince yourself that you've found the global optimum."

Or not. That last step is a potential doozy. Consider, for example, the design of a starfish. The task here for TreeMaker, in effect, is to pack five circles of equal radius inside a square. That's not such a hard problem. But what if your starfish were a 10-legged mutant? For more than a few dozen circles, mathematicians still aren't sure what the best packings are. (The first unsettled case is 28 circles. The only larger case that's been settled is 36 circles, for which the 6 × 6 square lattice packing is provably best. With 49 circles, there's a nonlattice arrangement that does better.)

For purposes of origami, however, TreeMaker's local maximum comes close enough, Lang says. "It's not perfect, but it's orders of magnitude better than the alternative!"

Applications

Lang's personal penchant is for origami insects, in part because they call for numerous long, skinny appendages, features that long eluded origami designers. (One origami book from the 1960s, Lang points out, told folders they would need to use two sheets of paper to make a six-legged creature.) He brought a number of beetles to the meeting, and a couple of scorpions. He also displayed a life-sized fish, complete with fins and scales. But one of his recent designs is decidedly nonorganic: the folding pattern for a space-based telescope.

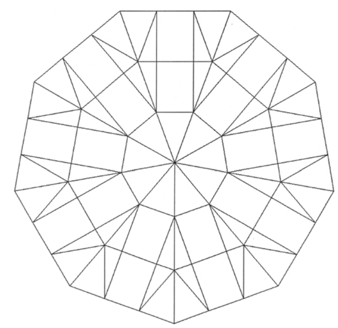


Figure 2. *God's monocle? One of the proposed crease patterns for a prototype space telescope, 5 meters in diameter.*

The origami telescope was described by Roderick Hyde and Shamasundar Dixit of the Lawrence Livermore National Laboratory, where the project is being developed. The basic idea, Hyde and Dixit explain, is to put huge telescopes, up to 100 meters in diameter, into orbit, for such purposes as the direct observation of extra–solar system planets. (The Hubble telescope, for comparison, is a mere 2.4 meters across.) But there's an obvious sticking point: Even at 5 meters, much less 100, an object with the shape of a manhole cover isn't going to fit into a spacecraft.

"That's where origami comes in," Hyde says.

Actually, origami is just one piece of an enormously complex engineering puzzle. The chief challenge, as usual, is achieving the required precision. A reflecting telescope, with its angstrom-order tolerances, seems to be out of the question. A standard refracting lens is equally unrealistic. The Lawrence Livermore designers opted for a Fresnel lens, which focuses light by diffraction.

Fresnel lenses are made from flat sheets of glass or plastic with sawtooth grooves in a circular pattern. You can find them at any math conference—they're used in overhead projectors. (Their original use, in the 19th century, was in lighthouses. On a sunny day, a large Fresnel lens can incinerate concrete.) Fresnel lenses are far more tolerant than reflectors of alignment errors: In a lens made of multiple sheets, the pieces can be misaligned by millimeters without compromising the focus. All you need to build a 100-meter telescope, then, is to keep the errors to around one part in ten thousand.

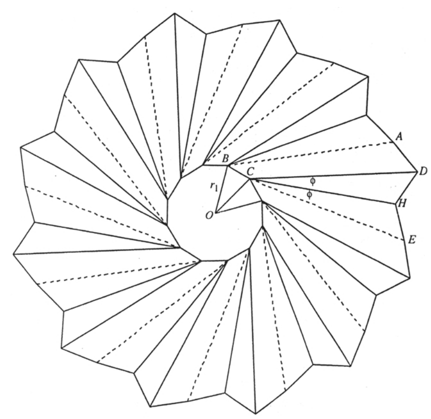


Figure 3. *Tomoko Fuse's origami design for a sturdy paper pot. (The dashed lines in this case are guidelines only. The solid lines are alternately mountain- and valley-folded. Assembly hint: Use paper clips to hold things in place until the pot "snaps" into place.)*

Hyde and Dixit described their group's current project, a 5-meter prototype. Weighing less than 100 pounds, the prototype "will be by far the largest lens in the world. It will be by far the lightest lens in the world," Hyde says. Targeted for completion in 2002, the prototype will consist of 81 glass panels hinged together in a spiderweb-like crease pattern designed by Lang (Figure 2).

"They had already come up with a bunch of ideas," Lang recalls, "but they wanted to make sure they'd examined everything before they did their design selection."

The Lawrence Livermore researchers demonstrated the basic design with a smaller, tabletop-sized Plexiglas model. When folded, the lens takes the shape of an open can. (Photos of the model being folded are available at http://db.uwaterloo.ca/~eddemain/photos/OSME_March2001/telescope/.) One of the important design criteria, to simplify the hinging, was that all the folds be single-layer—i.e., accordian pleating rather than the double folding used for, say, a business letter. Another, crucial criterion was that the folding/unfolding sequence (the lens will actually *spin* itself open) not require any flexing of the panels. The demo opens and closes like a glass-petaled daisy.

More down-to-earth applications of origami were introduced in talks by Rainer Hoffmann of EASi Engineering in Alzenau, Germany, and Tomoko Fuse, an origami designer in Nagano, Japan. EASi does computer-assisted engineering for the automotive industry. One of its specialties is software for simulating the deployment of airbags. Although "basic origami is not directly applicable," Hoffmann says that origami design techniques have been used in complex three-dimensional models for the folding patterns of airbags, which need to fit compactly in steering wheels, dashboards, and the curved roof structure of cars.

Fuse has designed—and patented—a sturdy paper pot that can be made from a single sheet of paper without the use of any glue. A key concept is that the folds creating the sides of the pot do not radiate directly from the center (see Figure 3). Unlike Lang's space telescope design, the paper pot must flex as it's being folded and, ultimately, "snaps" into position. This is the source of its strength: You can easily unfold the pot by pulling outward on one edge of its rim, but as that section of the pot unfolds, portions of other sections must flex slightly inward. Consequently, when an outward force is applied equally in all directions—as it is, say, if you've filled the pot with soup (or beer)—the sides stiffen because they are unable to flex inward. Fuse demonstrated the strength of the pots by using one of them to catch a ball that had been tossed in the air. Large, sturdy, glueless paper pots coated with a synthetic resin film or other water-repellent material could, she thinks, be an economically viable and environmentally friendly solution to the problem of disposable containers.

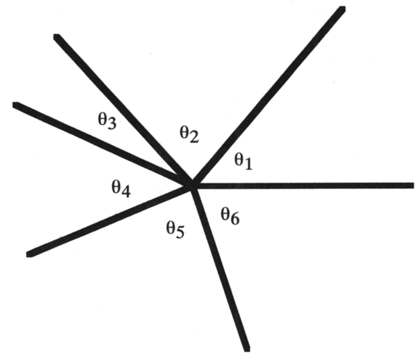


Figure 4. *Flat-foldability. For a crease pattern to be locally flat-foldable at a vertex, the number of creases meeting at the vertex must be even and the sum of every other angle must equal 180 degrees.*

Folding Made Difficult

Origami poses theoretical challenges as well. Chief among them is the issue of flat-foldability. Although most origami models are three-dimensional, one of the properties that origami designers typically strive for is the ability to take a 3D model and collapse it into a plane—for ease of transportation, if nothing else—without adding or undoing any creases. But what if all you're given is a crease pattern, with no clues as to which are mountain and which are valley folds? The task of distinguishing flat-foldable from non-flat-foldable patterns turns out to be surprisingly subtle.

What makes life difficult is not so much the number of folds as the number of vertices: points in the pattern at which two or more folds meet. There is a satisfactory answer, Hull says, only in the case of a single vertex. If N folds meet at a single vertex, with angles $\theta_1, \theta_2, \ldots, \theta_N$ between them (see Figure 4), a theorem proved by Toshikazu Kawasaki and Jacques Justin asserts that the pattern can be flat-folded if and only if N is even and the alternating sum $\theta_1 - \theta_2 + \theta_3 - \ldots - \theta_N = 0$. (The requirement that N be even is easy to understand: Every time you cross a crease, a flat-folded model flips between color up and color down, so by the time you get back to where you started, you must have crossed an even number of creases. The alternating-sum condition is also intuitively obvious with a little thought.)

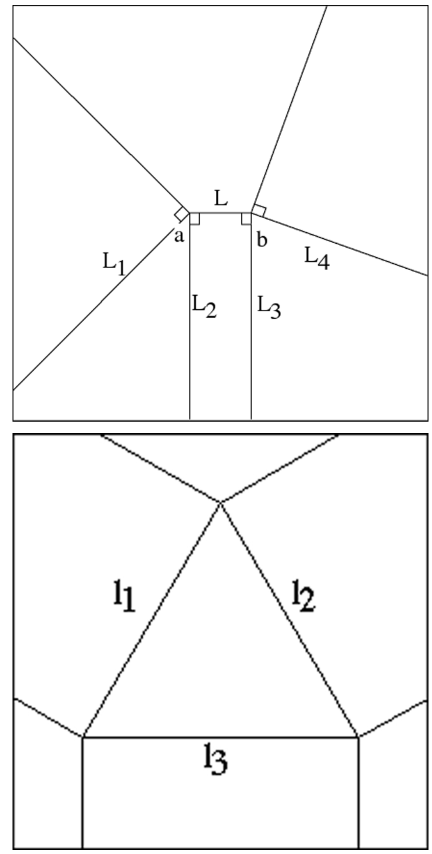


Figure 5. *Non-flat-foldability. Tom Hull has found crease patterns with two (top) and three (bottom) vertices that cannot be flat-folded, even though each vertex is locally flat-foldable. Give them a try! Note: In the two-vertex crease pattern, angle a = 45° and angle b = 70°.*

For single-vertex patterns satisfying the Kawasaki–Justin criterion, the next problem is to find the assignment(s) of mountain and valley folds that achieve flat-foldability. Here, a theorem of Justin and Jun Maekawa kicks in: The number of mountain and valley folds (call them M and V, with $M + V = N = 2n$) must differ by 2. If the angles are all equal, there's no other restriction: There are $2\binom{2n}{n-1}$ ways to make the assignment.

Hull recently proved a recursive formula for the number of assignments in general. Intriguingly, the formula requires a generalization of origami to conical paper (with the vertex of the crease pattern, of course, at the tip of the cone). The essence of the formula is to reduce the number of creases by eliminating small angles. If, for example, θ_2 is less than both θ_1 and θ_3, Hull's theorem then asserts that there are twice as many assignments for the crease pattern with angles $\theta_1, \ldots \theta_N$ as there are for the pattern with angles $\theta_1 - \theta_2 + \theta_3, \theta_4, \ldots \theta_N$. (This is where conical paper comes in: The new angles add up to less than 2π. The theorems of Kawasaki, Justin, and Maekawa still apply.) The general recursion formula includes the case of a string of equal small angles.

The "only if" parts of the Kawasaki–Justin–Maekawa theorems still apply when there's more than one vertex, but they are no longer sufficient to establish flat-foldability. The simplest counter-example has two vertices, each with four angles satisfying the alternating-sum condition (see Figure 5). And even when a pattern is flat-foldable, there might be just two ways to achieve it (if there's one way, there's always a second way—reverse mountain and valley, i.e., turn the paper over). This is the case for one of Hull's favorites, an octagon twist (see Figure 6). Of the 2^{24} possible mountain–valley assignments, 2^{16} satisfy the Justin–Maekawa condition for flat-foldability, but only two of them actually work.

It's hard enough to tell if a simple crease pattern is flat-foldable, although origami experts get good at it. But complicated patterns are a real challenge, and for good reason: Marshall Bern of Xerox Palo Alto Research Center and Barry Hayes of PlaceWare, Inc., have shown that the task is NP-complete. In 1996, they showed that the famous 3-SAT problem can be recast as a question about origami. The 3-SAT problem is concerned with the satisfiability of certain Boolean expressions. In Bern and Hayes's proof, these

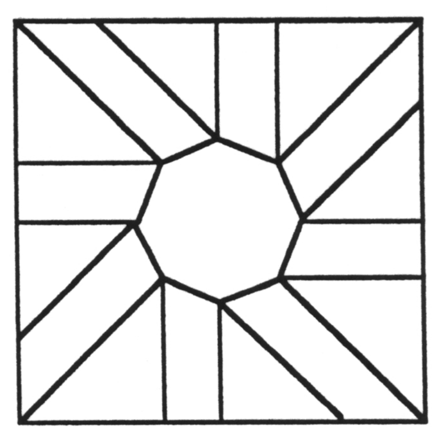


Figure 6. *Stop sign origami. There are just two ways to flat-fold this figure.*

logical formulas are turned into crease patterns in such a way that the Boolean expression is satisfiable if and only if the crease pattern is flat-foldable. Since 3-SAT is NP-complete (in fact, it's the original example of an NP-complete problem), so is flat-foldability.

More recently, Bern and Hayes showed that flat-foldability is NP-hard even when a correct assignment of mountains and valleys is given! Indeed, some problems become challenging only once a mountain–valley assignment has been made (see Figure 7). Roughly speaking, the remaining difficulty is to figure out the order in which to make the folds—different sequences put different parts of the paper at different levels locally, but these local stackings are often not globally consistent. In related work, Demaine and colleagues have shown that simple "map" folding, with mountain and valley folds on a rectangular grid, can be solved in linear time; with the addition of diagonal creases, however, the problem becomes NP-hard.

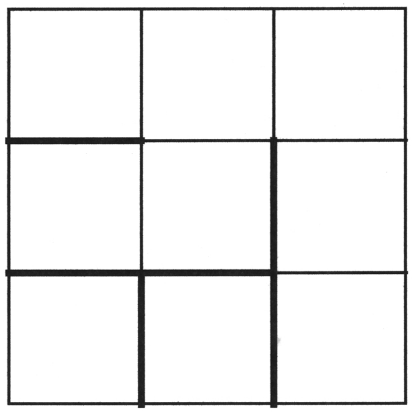


Figure 7. *Map folding. It's easy to fold a map if you're free to fold as you please. But how is it done when the mountain (thick) and valley (thin) folds are preassigned? (From "When Can You Fold a Map?", by Esther Arkin, Michael Bender, Erik Demaine, Martin Demaine, Joseph Mitchell, Saurabh Sethia, and Steven Skiena, http://db.uwaterloo.ca/~eddemain/papers/MapFolding/.)*

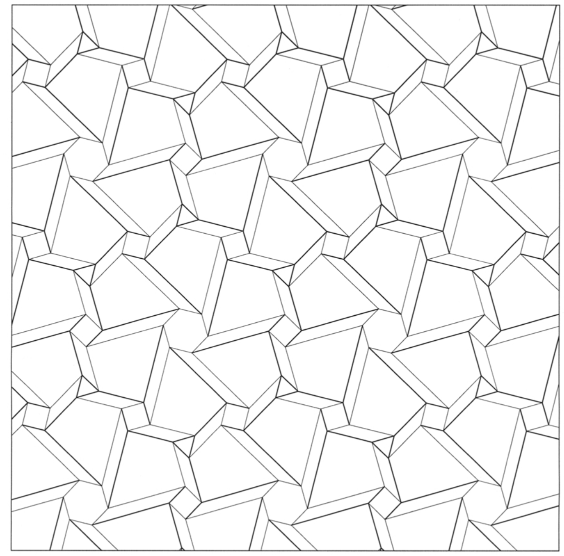


Figure 8. *Origami tessellation. Crease pattern generated by Tess, a computer program for designing origami tessellations.*

The complexity of flat-folding is hardly a deterrent to origami designers. For example, Alex Bateman, a biochemist at the Sanger Institute in England, has developed a computer program for designing flat-folding origami with (in principle) infinitely many creases. The program, Tess, creates crease patterns for origami tessellations (see Figure 8). The user is free to specify the underlying symmetry group and vary key parameters. (Tess is downloadable from Bateman's Web site, http://www.sanger.ac.uk/Users/agb/Origami/Tessellation/. Sample patterns are also available.) In theory, the tessellations are easy to fold: All you have to do is twist one part of the paper over another. The hard part is doing it in practice, without making the final product look like a crumpled mess.

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