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Iteration and its Consequences

Indra's Pearls: The Vision of Felix Klein. *By David Mumford, Caroline Series, and David Wright, Cambridge University Press, New York, 2002, 394 pages, $50.00.*

BOOK REVIEW

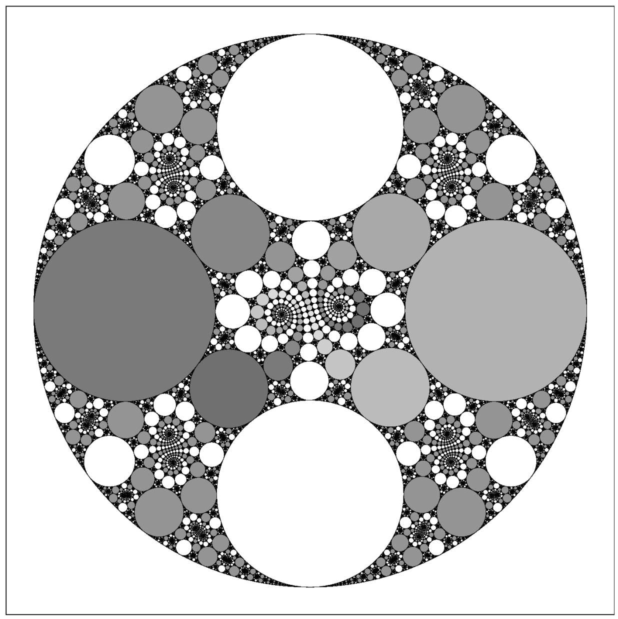
By Philip J. Davis

At a recent meeting of European historians of mathematics, I happened to raise the question "What is geometry?" I received no answer to my question during the sessions; the term is now so all-embracing that a public attempt at a definition would have been embarrassing. Later, one of the participants told me privately that for him geometry is the study of those figures that are left invariant under group actions and that the concept of *symmetry* is defined by such invariance. This, of course, is "the vision of Felix Klein" referred to in the subtitle of the book under review and known as Klein's *Erlanger Programm* (1872).

Since 1977, when Mandelbrot published his *Fractals: Form, Chance and Dimension,* geometry, to a very large constituency, has meant fractals and their production by iteration. The number of recent books dealing with fractals is enormous. Fractals have been explored and expounded on from numerous points of view: from theorem formation, from nature, from art and esthetics, as end products of mathematical experimentation and discrete dynamics, as mathematical models of natural phenomena, as ergodic and deterministic configurations.

The term is a hot one. Curricula at all levels are full of fractals (see, for example, Mandelbrot and Michael Frame's recent *Fractals, Graphics, & Mathematics Education*). Literary people have co-opted the term, with little concern for what Mandelbrot had in mind when he coined it. Fractals have hit Hollywood as part of virtual graphics. I don't yet own a fractal on a T-shirt but feel sure that I can get one.

Fractals are produced by iteration, and a very general and basic mathematical question is what happens when a process is repeated over and over again (i.e., iteration). In a primitive, anticipatory way, numerical analysts discovered fractals in their relentless search for convergence in iterative algorithms. They found convergence—to which they held fast and which they often tried to speed up. They found divergence—which they hated and discarded as useless, or tried, sometimes successfully, to force into convergence. They also found borderline cases, as when the iterates are all located on, say, the unit circle and are uniformly or equidistributed (*gleichverteilt*, pseudorandom). Such iterations find application in Monte-Carlo methods, simulation, and sampling.



A double-cusp group leads to interlocking spirals. Illustrations from Indra's Pearls.

The authors of *Indra's Pearls*—David Mumford, a Fields medallist now in the Applied Mathematics Division at Brown University, Caroline Series of Warwick University, and David Wright of Oklahoma State University—have made a marriage of two major geometric concepts: groups and fractals, joined in this marriage by iteration.

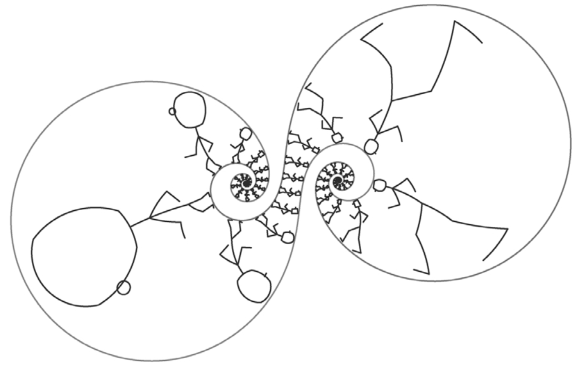
Mumford, Series, and Wright have clearly fallen in love with the group of Möbius maps ($T(z) = (pz + q)/(rz + s)$), ($ps - rq \neq 0$), its subgroups, and the iteration of two such maps, combined in all possible ways. Recall that a Möbius map in the complex plane is conformal, and sends lines and circles into lines or circles.

The theme of the book is to discover and explore, via iteration, the patterns that are simultaneously symmetric under the *two* Möbius maps. The text shows how initial choices can be made so that the "full range of complexity from relative order to total chaos will emerge." It traces paths from uninteresting fractal dust to more elaborate limit sets where, perched on the fragile boundary between order and chaos, the most striking visual patterns are often found.

Designating two noncommuting Möbius transformations by a and b and their inverses by A and B, the authors form the free group of all possible Möbius maps keyed as "words" that can be formed from the concatenation of these four letters (i.e., maps); an example is *aBBaBab*. There follows an extremely detailed study of what happens when one starts out with an initial configuration of circles and subjects it systematically to all the maps in this group. How a and b interact is critical. Depending on what one selects as the two initial generators and on certain initial circles de-termined by them and to be mapped, one arrives at circles completely packed inside an initial circle, circles within circles . . . within circles . . . displayed down to pixel level ("Indra's Pearls"). Such packings often show spiral configurations of various kinds.

The text tags by name many specific free groups, e.g., Fuchsian groups, Schottky groups, all developed from Möbius maps. A "roadmap" showing the interconnection between 15 of these groups appears on the final page. It is amazing what a rich set of figures, ideas, and as yet unsolved problems arise from the simple limitation of the generating transformations to Möbius. That complexity can arise from simplicity is a well-known phenomenon, prominent recently as the keystone, within the context of cellular automata, of Stephen Wolfram's *A New Kind of Science* (Wolfram Media, 2002; see review by James Case).

Getting down to the limit sets, we expect considerable complexity as the two Möbius maps leave no elementary measure invariant. Fractals emerge, and we recognize their familiar hallmarks: self-similarity at all scales—the microcosm in the macrocosm; fractal dust (question: can a noninteger Hausdorff dimension be identified as such by the naked eye?); and visual appeal, with figures that some people find beautiful or at least intriguing. All these sets have been produced at a level of detail that would not have been possible without computer graphics and that could not have been displayed without state-of-the-art printing techniques.



Helping to clarify the properties of maps considered in the book under review is Dr. Stickler, shown here taking a spiral walk.

The exposition in *Indra's Pearls* is relaxed, and its level will be agreeable and rewarding to the nonspecialist. There are included, for example, discussions of the elementary properties of Möbius maps, their representation in matrix form, the algebra and the programming strategies for dealing with word trees. The text walks the reader step by step through the succession of images as they develop, often employing amusing stick figures (Dr. Stickler) to produce a clearer understanding of what the maps do. At a more advanced level, as when the authors wrap tilings into surfaces, some familiarity with topology and its goals would be useful. The authors have also provided biographical and historical sketches, algorithms and pseudocode, suggested projects, and references to Web sites where one can watch animated fractals.

The production of the book leaves nothing to be desired. It is splendid. Printed entirely on glossy paper, with practically all of the many figures in glorious color, the book has a number of admirable design features: large type and wide margins wherein references are given and occasional comments (often quite talky) are made. CU Press has done a beautiful job, and David Tranah of the CU Press deserves special commendation for his role in pulling out all the stops.

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"*In the beginning, then, was the repetition.*"—*Terry Eagleton**

Just as one can take transformations forward, e.g., *a*, *aa*, *aaa*, . . . , or backward, *A*, *AA*, *AAA*, . . . , allow me first to go backward in mathematical time, to the emergence of the exponential notation for successive multiplications. Both positive and negative exponents appear to have been introduced by the German mathematician Michael Stifel (1487–1567). Some say that Stifel just missed inventing logarithms.

Stifel was a monk who later became a follower and correspondent of Luther and a Protestant preacher. In his younger days, by manipulating triangular numbers (developed by iteration) and by identifying letters with the numbers, Stifel predicted that the end of the world would take place at 8 A.M. on October 19, 1533. This naturally caused a stir in his small village, and once the date had passed, he found himself in deep hot water for his miscalculation. Extricating himself with the aid of Luther, Stifel then went on to do good work in algebra (*Arithmetica integra*, 1544).

Taking the process of iteration forward beyond the present day, Wolfram's *A New Kind of Science*, a book that has been making a stir in our own village, predicts the end of formulas and functions. They are to be replaced by the iteration of simple rules applied to cellular automata. On this basis, revolutionary paradigm switches are forecast for all scientific fields. The future of iteration is thus assured.

In a Buddhist legend, which provided the authors with their exotic title,

> "The heaven of the great god Indra is a vast and shimmering net. At each intersection of its . . . threads is a reflecting pearl. In the glistening surface of each pearl are reflected all the other pearls. In each reflection again are reflected all the infinitely many other pearls, so that by this process reflections of reflections continue without end."

Without invoking theological or teleological positions, it might be said that a good fraction of life consists of iterative actions—if only tying your shoelaces every morning. Whether this gets you a calm, harmonious or a chaotic life depends on the group that you are in.

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