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An Accidental Participant in the Golden Age of German Mathematics

Constantin Carathéodory: Mathematics and Politics in Turbulent Times. *By Maria Georgiadou, Springer, Berlin, 2004, xxviii + 651 pages, $99.00.*

BOOK REVIEW

By James Case

A mathematical biography of Constantin Carathéodory seems long overdue. As a partial solver of the Hilbert problem concerning axioms for thermodynamics, the designer of the "royal road" to the calculus of variations, a founding father of geometric function theory, the first visiting lecturer of the American Mathematical Society, the author of several gold-standard graduate-level texts, a longtime editor of the *Annals of Mathematics*, and the chair of the very first Fields Medal committee*, his reputation among German mathematicians of the 1920s and 1930s was exceeded only by Hilbert's. Yet he is little remembered today, save by specialists in the fields he helped establish.

In 1994, Greece issued a stamp recognizing Carathéodory's contributions to mathematics.

Carathéodory was born in Berlin, on September 13, 1873, to a Greek family prominent in the Ottoman Empire. Between 1840 and 1912, at least fifteen Greek diplomats represented the empire in such capitols as Athens, Vienna, London, Rome, Berlin, Brussels, The Hague, and Washington; Carathéodory was related to ten of them. Other members of the family, including his great-uncle Konstantin Carathéodory, his uncle Telemachos Carathéodory, and especially his cousin Ioannis Aristarchis (aka James Bey), were among the empire's leading engineers. Late in life, Constantin found it impossible to account for his family's extraordinary record of sustained achievement save in terms of the "unique historical events" that had, for so many generations, afforded them seemingly endless opportunity.

The family moved often during Carathéodory's early childhood, with extended stays in Berlin, Cannes, Constantinople, and San Remo—interrupted by periodic visits to relatives in Greece and Asia Minor—before settling in Brussels, where his father served as the Ottoman ambassador to Belgium between 1875 and 1900. There Constantin attended a variety of (French-language) schools, before enrolling as a student of artillery and engineering at the Belgian Military Academy in 1891. Quartered in Spartan barracks, and required to spend long hours in the saddle and on the parade ground, qualified cadets, Carathéodory among them, nonetheless availed themselves of the extensive technical training offered at the academy. Though he considered his calculus course antiquated, Carathéodory was more than satisfied with the instruction he received in mechanics, probability, astronomy, geodesy, and thermodynamics. His lifelong fascination with descriptive geometry, which 19th-century engineers were taught to use as a sort of analog computer, began during his years at the academy.

After graduating in 1895, Carathéodory traveled for two years in Europe and Asia Minor, pausing at one point to assist his cousin Ioannis Aristarchis in the design of a road network for the island of Samos. In 1897, as an engineer in the British colonial service, he traveled to Egypt to take part in dam construction along the Nile. Because the annual flooding brought construction to a virtual standstill between September and January, he had ample time to continue his study of mathematics. He had brought along a number of books on the subject, including Jordan's *Cours d'Analyze* and a book by George Salmon on conics.

In the summer semester of 1900, Carathéodory enrolled in the Friedrich Wilhelm University in Berlin, later renamed for Alexander von Humboldt. His decision to abandon what seemed a promising career in engineering outraged most of his family members, who doubted that mathematics could lead to anything more lucrative than high school teaching. In later life, Carathéodory conceded that he had shared a number of their doubts, but yearned for the meaning that mathematics alone seemed to bring to his life. He chose Berlin over Paris because, having fewer relatives in Berlin, he felt that he would be less distracted there. So it was that, due to a mere accident of expatriate geography, he became a participant in the Golden Age of German mathematics.

In addition to lectures in pure mathematics by H.A. Schwartz, Fuchs, and Frobenius, he began attending lectures by Planck on mechanics and Maxwellian electrodynamics, by Julius Bauschinger on celestial mechanics, and by Carl Friedrich Stumpf on symbolic logic. Within weeks, he formed close friendships with Erhard Schmidt and Leopold Fejér, both of whom were enthusiastic participants (along with Friedrich Hartogs, Paul Koebe, Oliver Kellog, and others who subsequently rose to prominence) in Schwartz's colloquium. The friendships he formed that summer helped shape the whole of his career.

In the early years of the 20th century, the center of German mathematics moved from cosmopolitan Berlin to provincial Göttingen. Though the latter already had a formidable mathematical tradition—mainly because Gauss, Dirichlet, and Riemann had

* The other members were Henri Cartan, George Birkoff, Francesco Severi, an Italian geometer, and Teiji Takagi, a Japanese mathematician.

worked there—it did not become a magnet for foreign visitors until the late 1890s.

Carathéodory's friend Erhard Schmidt left Berlin for Göttingen early in 1901, and was so filled with excitement during a Christmas visit that Carathéodory soon followed him. Arriving in Göttingen in the summer of 1902, he was delighted to reunite with Schmidt and Fejér, and to make the acquaintance of Ernst Zermelo, Max Born, Otto Blumenthal, William H. and Grace Chisholm Young, Herman Minkowski, and (with surprising ease) both Klein and Hilbert. He became particularly close to Klein, who found Carathéodory's facility with Monge's descriptive geometry—acquired first at the Belgian military academy and later in the field of practical engineering—quite fascin-ating. Their friendship grew closer with the passage of time and continued until Klein's death.

Contemporaries described Carathéodory—who stood only about 5 ′ 7″—as tall, impressive, and aristocratic in appearance, exuding confidence and in command of many languages. He soon joined, first as a guest and later as a full-fledged member, a fashionable group of young scholars from all faculties, to which Zermelo and Schmidt already belonged. Members of the group adhered to a strict though unwritten code of conduct, and they convened regularly at the Black Bear Pub in the old town of Göttingen for Bavarian beer and conversation. A student could attend only as the occasional guest of a member, and in the summer of 1904, having defended his thesis on "discontinuous solutions to problems in the calculus of variations," Carathéodory began to invite the physicist Max Born to join the group.

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That fall, Carathéodory returned to his "native" Greece in search of employment. Not yet aspiring to a position in the nation's only university, he sought a teaching position at a military or naval academy and was bitterly disappointed when offered nothing more than a job teaching the Greek language at a school in the provinces. Returning to Germany, he submitted a second thesis (Habilitation) to the Göttingen faculty in March 1905, thereby gaining authorization to lecture during the tenth semester of his studies. The title of his inaugural address, "On Length and Area," was resurrected in December 1949 for a public lecture in Munich, which turned out to be his final public appearance. He died unexpectedly on February 2, 1950.

Also in 1905, Carathéodory published a note in the *Comptes Rendus* sharpening Picard's theorem to the effect that, just as the range of e^z omits only the value 0, the range of any entire function can omit at most one complex value. His brief note improved on a result of Edmund Landau and attracted a great deal of attention at a time when his name was unknown outside Germany. It was followed by a rapid sequence of fundamental results on complex analysis, conformal mappings, and geometric function theory.

During the summer of 1908, Carathéodory began to lecture at the Rhine University in Bonn, where he was soon promoted to the rank of professor. In November he completed his work on the axiomatic foundations of thermodynamics, which was published in the *Annals of Mathematics* the following year. Whereas his contemporaries found it altogether satisfactory and credited him with the solution of the part of the Hilbert problem asking for an axiomatization of thermodynamics, his formulation has never made its way into the physics curriculum and has been harshly criticized by certain modern scholars. Georgiadou quotes C. Truesdell's claims that the work represents "axiomatization for axiomatization's sake" and fails to distinguish between "supplies of heat at different temperatures." As the late Gian-Carlo Rota wrote (*Indiscrete Thoughts*, 1997), "It is a standing bet between physicists and mathematicians that thermodynamics cannot be axiomatized."

In 1909, Carathéodory married his aunt Euphrosyne Carathéodory—11 years his junior—in Constantinople. The marriage was by all accounts a happy one and soon resulted in the birth of a son, Stephanos, and daughter, Despina. The former contracted polio at an early age and seems to have lived a rather restricted life. In marrying a relative, Carathéodory was following a policy of long standing within his family, designed to increase their combined political and financial power within the Ottoman Empire.

On his return to Germany, Carathéodory began to involve himself in educational policy and politics, moving quickly from one university to another as positions became available. Between 1910 and 1918, he held "budgeted professorships" at universities in Hannover, Breslau, Göttingen, and Berlin, while lecturing at several others. By including pictures of the houses he and his family inhabited during this time, the book manages to convey a sense of the ease and comfort in which university professors could expect to dwell in the Kaiserreich.

German academics belonged to the so-called *Bildungsbürgertum*, a segment of the upper middle class that enjoyed both rank and respect in the class-conscious prewar society. Only the "turnip winter" of 1917—during which Carathéodory supplied his children with milk by keeping goats in the basement—seriously interrupted the family's comfortable life.

His treatise on real functions, published in 1918, was the first presentation of the still-developing Lebesgue theory in textbook form. Though meant as a text, it contained a number of original results, including the fact that the right-hand side of the equation $\dot{x} = f(x,t)$ need only be bounded and measurable in t, for each fixed x, in order to justify the usual conclusions on the existence and uniqueness of solutions. Though it seemed a meaningless generalization at the time, it turned out to be just the generalization needed for the modern theory of optimal control, formulated during the 1950s.

Greece expanded at the end of the Balkan War of 1912–1913 to include sizeable parts of modern Albania, Serbia, and Bulgaria, and again in 1919, by acquiring parts of modern Turkey following the breakup (during World War I) of the Ottoman Empire. To the negotiators of the Versailles treaty, the acquistions seemed to be justified by the fact that much of the territory in question had been inhabited by Greeks since the time of the Trojan War. Among the jewels of the newly acquired territory was the city of Smyrna (now Izmir), situated in the part of the Turkish mainland nearest Athens.

The newly re-elected prime minister of Greece, Eleutherios Venizelos, having known Carathéodory for twenty-five years, promptly appointed him to organize a new university at Smyrna. Without hesitation, he accepted the post and relocated his family to the site before returning alone to Europe in quest of the faculty, library, and scientific apparatus the new institution would need. The results were barely in place when, on September 9, 1922, the forces of Kemal Attaturk completed their conquest of Anatolia by occupying Smyrna. Carathéodory was forced to flee, with his family, books, and manuscripts, as well as the university library and much of its apparatus. Attaturk's conquest ended any serious talk of a "Greater Greece" encompassing all her territories from Homeric times.

After escaping to Athens, Carathéodory moved into his sister's house, where he remained until May 1924. He then received an offer to succeed Ferdinand von Lindemann (who proved π to be transcendental, thereby demonstrating the impossibility of squaring the circle) at the Ludwig Maximilian University in Munich. There he remained, with time off for extensive travel to the U.S. and Middle East, until he retired at the age of 65 in 1938. Anxious though he had been to help usher Greece into the modern age, he found his years there (especially his reception by the entry-level students he was assigned to teach in Athens) discouraging in the extreme. In his memoirs he described the failure of his efforts to turn Greece into a modern center of scientific excellence as the main disappointment of his otherwise eminently satisfactory career.

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In 1935, shortly before his retirement, Carathéodory published his two-volume treatise on first-order partial differential equations and the calculus of variations. It presented, among other things, his highly original royal road to the latter. That road readily extends to problems in optimal control theory, such as the following:

$$\begin{aligned}&\text{maximize}_{u(t)}\ J=\int_0^T L\big(x(t),u(t)\big)dt\\ &\text{subect to: } \dot{x}(t)=f\big(x(t),u(t)\big);\\ &x(0)=\xi \text{ and } \forall t, u(t)\in U, x(t)\in X,\\ &\text{and } x(T)\in\Omega\subset\partial X,\end{aligned}\qquad \text{(P}$$

for given sets U, X, and Ω in Euclidean spaces of appropriate dimension, where T denotes the earliest instant t at which $x(t)\in\Omega$.

Problems in the calculus of variations correspond to the special case $f(x,u)\equiv u$ of (P). Carathéodory's royal road begins with the observation that, if (P) is simple in the sense that the conditions

$$\begin{aligned}&L(x,u)\le 0, \forall (x,u)\in X\times U\\ &L\big(x,\phi^*(x)\big)=0, \forall x\in X\end{aligned}\tag{1}$$

are satisfied for some function $\phi^*(x)$, then ϕ^* is an optimal "feedback control" for (P) because it causes J to vanish, while no rival feedback control can impart a more desirable (i.e., larger) value to J. The desired optimal "control history" is then $u^*(t) = \phi^*(x(t))$. Carathéodory separates problems of the form (P) into equivalence classes in such a way that equivalent problems share a common optimal feedback control; he observes that solving the so-called Hamilton–Jacobi equation, which he was apparently the first to write in the form

$$\max_{u\in U}\left\{L(x,u)+<\nabla S, f(x,u)>\right\}=0,\tag{2}$$

for the (unknown) function $S(x)$, leads to a second problem—equivalent to the first—that is simple in the foregoing sense.

The final chapters of Georgiadou's book detail Carathéodory's travels in the United States as the first star in the galaxy of AMS visiting lecturers, his role (extraordinarily prominent, due to both his eminence and his multilingual capabilities) in the International Congresses of Mathematicians for 1932 and 1936, and the illnesses that led to his own and his wife's deaths soon after World War II. An entire chapter is devoted to his decidedly controversial role during the Third Reich. Many thought he could and should have done more for the mathematicians of Eastern Europe—especially those in Poland—many of whom eventually died in concentration camps. Yet it remains unclear what more he might have done, short of sharing their fate.

This is not a particularly entertaining book, nor an easy one to read. The need to provide background information about the Ottoman Empire and its breakup during World War I inevitably slows the pace. Yet Georgiadou's scholarship is impressive. She has interviewed specialists in the branches of mathematics to which Carathéodory contributed most and summarized what must now be considered mature opinions of his work. Finally, by consulting what still exists of the principals' correspondence and official records, she reveals much about the inner workings of the mathematical community within which Carathéodory lived and worked.

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