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Buried Treasures

1089 and All That: A Journey into Mathematics. *By David Acheson, Oxford University Press, Oxford, UK, and New York, 2002, 178 pages, $24.95.*

BOOK REVIEW

By James Case

The title of this pocket-sized volume reflects the author's hope of capitalizing on the enduring popularity (mainly in England) of a 1930 book titled *1066 and All That*. That minor classic, by W.C. Sellar and R.J. Yeatman, purports to explain English history as a typical country squire might have done. Generations have found the satire so hilarious that (again in England) the book has never gone out of print. The book under review has little in common with the older one, being well written but not deliberately funny, and having to do with mathematics rather than history.

The figure 1089 derives from the fact that, if one begins with any three-digit number whose first and last digits differ by at least two, then proceeds to (a) interchange the first and last digits, (b) subtract the smaller of the resulting numbers from the larger, (c) interchange the first and last digits of the difference, and (d) add the new result to the previous one, the sum is always 1089. This is revealed in the first chapter, though the explanation is deferred to the fourth, "The Trouble with Algebra." Acheson learned the trick from a children's magazine when he was but ten years old, and never again—despite the boredom induced by his grammar school curriculum—doubted that mathematics can be fascinating. Although the fateful magazine presented other conjuring tricks, the mystery and surprise of the 1089 stunt elevated it above all the others in his youthful esteem.

The rest of the book leads the reader to treasures buried just below the surface in other corners of the field. He separates his treasures into three main categories: Wonderful Theorems, Beautiful Proofs, and Great Applications. To wish that he had cited "important practical applications" rather than great ones would be to quibble with his otherwise impeccable expository judgment.

The second, fourth, and sixth of the book's 16 chapters devote about ten pages each to geometry, algebra, and calculus. More interesting by far is chapter 3: "But . . . that's Absurd . . . ." Having ended chapter 2, "In Love with Geometrie," with an example in which the obvious conjecture turns out to be wrong—thereby stressing the need for mathematical proof—he devotes the whole of chapter 3 to proof by contradiction. Without displaying Euclid's occasionally excessive enthusiasm for the method, or mentioning Brouwer's aversion to it, he illustrates its use by solving the Königsberg bridge problem, demonstrating the infinitude of primes, and explaining its relevance for Fermat's last theorem.

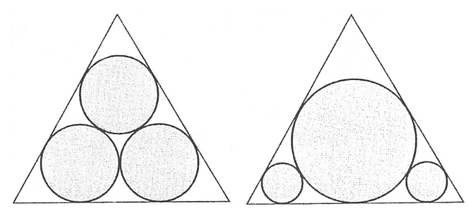

Figure 1. *Malfatti's mistake, undetected for more than a century: The circles in the triangle on the left cover less than 73% of the area, while those in the one on the right cover almost 74%. Figures from* 1089 and All That.

Equally accessible to the inexperienced reader is chapter 11, "Great Mistakes." In it, Acheson makes the point that mistaken proofs can remain undetected for extended periods of time. In 1803, for instance, Malfatti asked how three non-overlapping circles should be placed within a given triangle to make the area covered as large as possible. The answer, he declared, was that each circle should touch two sides of the triangle, as well as both other circles. It was not until 1930 that someone noticed an error in Malfatti's solution as it applies to equilateral triangles (see Figure 1).

Howard Eves pointed out later that Malfatti's solution is incorrect in numerous other cases, and then, in 1967, Michael Goldberg demonstrated that it is never correct—the optimal configuration always has one of the forms shown in Figure 2, in which one of the circles touches all three sides of the given triangle.

Another example is Kakeya's problem, named for the Japanese mathematician who posed it in 1917. It asks for the smallest area in which a line segment of unit length can be turned end for end. The simplest approach is simply to rotate the segment about its midpoint through 180°, which causes it to sweep out a circle of radius ½ and area $\pi/4 = 0.78. . . .$ A better idea is to rotate the segment through 60° about an endpoint, letting T denote the (unique) equilateral triangle of unit altitude that contains the circular sector just swept out. If the segment is then slid as far as it will go along the side of T on which it comes to rest, and rotated through another 60° about its other end, it will come to rest along the third side of T. Repeating the process one more time brings the segment back to its original position, with ends reversed. The area swept out (much of it three times over) is then that of T, which is only $1/\sqrt{3} = 0.58. . . .$

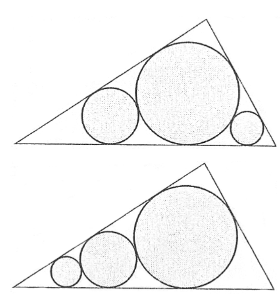

Figure 2. *Malfatti's solution is* never *correct: As shown in 1967 by Michael Goldberg, the optimal configuration takes one of the forms shown here.*

An even better idea involves the hypocycloid traced out by a point on the edge of a circle of radius ¼ rolling around the inside of a circle of unit radius, as indicated in Figure 3. Sliding the segment along each arc in turn brings it back to its original position with ends reversed. Moreover, the area swept out is only $\pi/8 = 0.39 . . . .$ This was believed to be the best possible result until 1927, when Besicovitch demonstrated that the problem actually has no solution. Whereas every orientation-reversing route must sweep out some area, routes can be found that sweep out no more than a preassigned e. Even acknowledged experts make mistakes.

The book includes chapters on π and on e, as well as on the limits of sequences and series, optimization, and (good) vibrations. All follow the same general pattern, proving a little of what is easily proven before attempting to seduce the inexperienced reader with unexplained mysteries. Acheson is anxious to explain, in his chapter on π, that π forms a part of the answer to all manner of problems that might seem unrelated to circles.

Following chapters on differential equations, stability of motion, chaos, and catastrophe comes a chapter titled "Not Quite the Indian Rope Trick." After a discussion of double, triple, and multiple pendulums, it proceeds to the observation that inverted pendulums are unstable. This leads into a discussion of Acheson's own most famous discovery, according to which an inverted multi-pendulum can be stabilized in the vertical position merely by moving the bottom pivot up and down with the proper frequency and amplitude. A fuller account of this work is available in an earlier (and slightly more advanced) book [1].

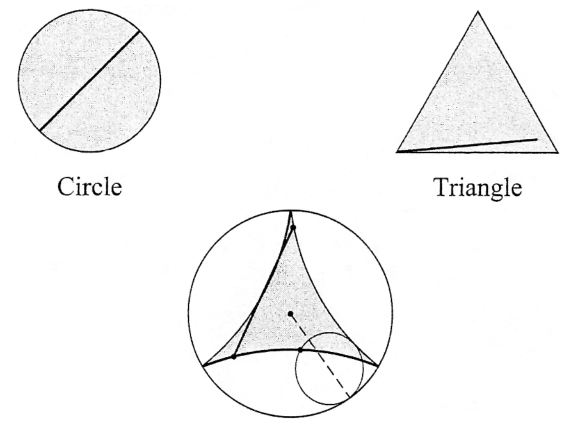

Figure 3. *Until 1927, the hypocycloid traced by the point on the edge of the circle was believed to be the best possible solution to Kakeya's problem (find the smallest area in which a line segment can be turned end for end).*

So fascinated did Acheson become with his finding that, after constructing computer simulations of inverted double and triple pendulums, he persuaded a leading experimentalist to help him construct an actual working model. While the inverted double pendulum was soon up and running, a 50-cm triple pendulum proved trickier. Not until the pivot began to vibrate up and down through 2 cm at about 40 cycles per second did the motion appear to stabilize. Once that was accomplished, however, the contraption proved unexpectedly stable. Even when pushed as much as 40 degrees out of plumb, it would eventually "wobble back to the upward vertical." In October 1995, he and a colleague demonstrated their version of the "Indian Rope Trick" on BBC. Though they received only three or four of the fifteen minutes of fame Andy Warhol promised us all, they thoroughly enjoyed the experience. Many more will enjoy Acheson's little gem.

References

[1] D. Acheson, *From Calculus to Chaos: An Introduction to Dynamics*, Oxford University Press, Oxford, UK, and New York, 1997. (Reviewed in *SIAM News*, Vol. 32, No. 6, 1999; http://www.siam.org/siamnews/07-99/julytoc.htm.)

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