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Birth of a Literary Genre

The Great Equations: Breakthroughs in Science from Pythagoras to Heisenberg. *By Robert P. Crease, W.W. Norton, New York, 2008, 224 pages, $25.95; paperback, 2010, 315 pages, $16.95.*

In Pursuit of the Unknown: 17 Equations that Changed the World. *By Ian Stewart, Basic Books, New York, 2012, 352 pages, $26.99.*

The Universe in Zero Words: The Story of Mathematics as Told Through Equations. *By Dana Mackenzie, Princeton University Press, Princeton, New Jersey, and Elwin Street Productions, London, UK, 2012, 224 pages, $27.95.*

BOOK REVIEW

By James Case

If three quite different books by quite different authors with quite similar titles can be said to constitute a new literary genre—call it the "great equations" genre—it may be time for a birth announcement. Each of the three books reviewed here undertakes to explain an important part of intellectual history through the medium of equations.

Two of the authors are well known to the SIAM community. Ian Stewart, a professor emeritus of mathematics at Warwick University, is among the most prolific popular mathematics writers of all time. Dana Mackenzie, who received the JPBM Communications Award last year (following its presentation to Stewart by more than a decade), has written for *Discover*, *Smithsonian*, *Science*, *New Scientist,* and *SIAM News*.

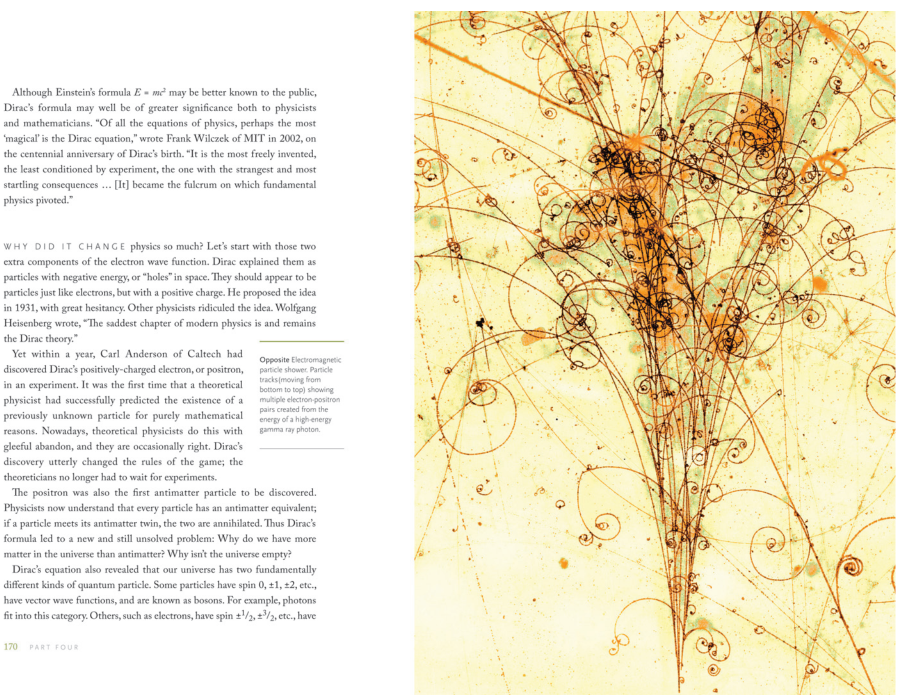
Robert Crease, who chairs the philosophy department at Stony Brook University, is the author of several popular science books. He is also the official historian of the Brookhaven National Laboratory.

The three authors make use of a common format. A sequence of "equation chapters"—each devoted to a single equation or system of equations—is preceded by a brief introduction and/or preface and followed by a single chapter of "conclusions." Each equation chapter begins with a display of the featured equation(s), followed by a discussion of meaning and content, and a discourse on historical significance. Crease offers ten such chapters, and relegates his commentary to brief "Interludes" separating his substantive chapters. Stewart presents 17 equation chapters, and Mackenzie, 24.

All three authors devote entire chapters to eight equations deemed great by consensus: Pythagoras's $a^2 + b^2 = c^2$, Newton's $F = ma$ and $F = GMm/r^2$, Euler's $e^{i\pi} + 1 = 0$ and $V - E + F = 2$, Einstein's $E = mc^2$, Maxwell's equations of electromagnetism, and the fundamental equation of quantum mechanics. Crease and Stewart take the latter to be Schrödinger's equation, while Mackenzie opts for Dirac's. Each author includes at least two equations omitted by the other two.

Crease—more concerned with the equations of physics than those of pure mathematics—considers only two equations that are missing from the lists of the other two authors: Heisenberg's uncertainty principle and Einstein's field equations of general relativity. Both Mackenzie and Stewart discuss general relativity in chapters on $E = mc^2$. Stewart, in particular, extends his discussion to include current events in cosmology, beginning with the discovery that the universe appears to be expanding at an increasing rate. Cosmologists are currently unable to explain this apparent fact without assuming the presence of large quantities of unobservable (dark) matter and energy in the otherwise empty space between observable (luminescent) heavenly bodies. Even with these powerful hypotheses, Stewart writes, the available facts are proving difficult to explain.

After describing the content of the uncertainty principle, Crease chronicles the dispute it occasioned between Heisenberg and Bohr concerning the manner in which quantum mechanics is to be interpreted, as well as the uneasy truce between them that became the "Copenhagen interpretation."


Although Einstein's formula $E = mc^2$ may be better known to the public, Dirac's formula may well be of greater significance both to physicists and mathematicians. "Of all the equations of physics, perhaps the most 'magical' is the Dirac equation," wrote Frank Wilczek of MIT in 2002, on the centennial anniversary of Dirac's birth. "It is the most freely invented, the least conditioned by experiment, the one with the strangest and most startling consequences … [It] became the fulcrum on which fundamental physics pivoted."

WHY DID IT CHANGE physics so much? Let's start with those two extra components of the electron wave function. Dirac explained them as particles with negative energy, or "holes" in space. They should appear to be particles just like electrons, but with a positive charge. He proposed the idea in 1931, with great hesitancy. Other physicists ridiculed the idea. Wolfgang Heisenberg wrote, "The saddest chapter of modern physics is and remains the Dirac theory."

Yet within a year, Carl Anderson of Caltech had discovered Dirac's positively-charged electron, or positron, in an experiment. It was the first time that a theoretical physicist had successfully predicted the existence of a previously unknown particle for purely mathematical reasons. Nowadays, theoretical physicists do this with gleeful abandon, and they are occasionally right. Dirac's discovery utterly changed the rules of the game; the theoreticians no longer had to wait for experiments.

The positron was also the first antimatter particle to be discovered. Physicists now understand that every particle has an antimatter equivalent; if a particle meets its antimatter twin, the two are annihilated. Thus Dirac's formula led to a new and still unsolved problem: Why do we have more matter in the universe than antimatter? Why isn't the universe empty?

Dirac's equation also revealed that our universe has two fundamentally different kinds of quantum particle. Some particles have spin 0, ±1, ±2, etc., have vector wave functions, and are known as bosons. For example, photons fit into this category. Others, such as electrons, have spin $\pm^1/_2, \pm^3/_2$, etc., have

Opposite Electromagnetic particle shower. Particle tracks(moving from bottom to top) showing multiple electron-positron pairs created from the energy of a high-energy gamma ray photon.

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Electromagnetic particle shower. Particle tracks (moving from bottom to top) showing multiple electron–positron pairs created from the energy of a high-energy gamma ray photon. Two-page spread from The Universe in Zero Words*; courtesy of Elwin Street Productions.*

Stewart includes seven great equations omitted by the other two, and Mackenzie presents 14. That is in part because Mackenzie begins nearer the origins than the others, his first two great equations being $1 + 1 = 2$ and $1 - 1 = 0$. In the chapter devoted to the former, titled "Why We Believe in Arithmetic," he argues that our belief is mainly empirical. The results of arithmetic computation would not be accepted if they did not conform with experience involving sheep, goats, shekels of silver, yards of cloth, and so on.

It was not until the late 16th century, Mackenzie adds, that European authors began to substitute symbols for words like sum, difference, and product. The now ubiquitous equals sign made its European debut in a book called *The Whetstone of Wytte*, published in 1557 by one Robert Recorde. Although it had been known for millennia that $1 + 1 = 2$, the equation was probably not written in symbolic form until some time in the 16th century. The subtler $1 - 1 = 0$ took even longer, due to the resistance with which the concept of zero was met in western Europe.

Several of the equations considered, such as Crease's $\Delta x \Delta p \geq \hbar/4\pi$, Stewart's $dS \geq 0$, and Mackenzie's

$\pi = 3.1415926535...$ and $2\uparrow\aleph_0 = \aleph_1$, would not ordinarily be thought of as equations. Yet they help the authors describe important events in the history of mathematics briefly and understandably. (Here S stands for entropy, as in the second law of thermodynamics, and $\hbar$ denotes Planck's constant.)

Crease begins his chapter on the Pythagorean theorem by rehearsing the proof of a special case of it that Socrates coaxed out of the untutored slave boy Meno (according to Plato's dialogue of that name). He then describes the dramatic effect of a chance encounter with a copy of Euclid's *Elements*—left open at the page containing Pythagoras's result—on the life and subsequent career of Thomas Hobbes, and the equally important influence of the theorem on the careers of Descartes, Einstein, and others. Indeed, Crease speculates, the theorem appears to epitomize—for Hobbes, Plato, and many others—the very *idea of mathematical proof*. By contrast, Stewart emphasizes the role of the theorem in the development of trigonometry, cartography, coordinate geometry, and the knowledge that the earth is round, and Mackenzie focuses on the contribution of the theorem to the discovery of irrational numbers, and the failure of the Chinese and Babylonians (both of whom knew the theorem before Pythagoras) to discover that $\sqrt{2}$ is irrational.

Mackenzie's chapter "The Great Explorer: Euler's Theorems" describes what Euler accomplished while maintained at various courts of 18th-century Europe. In addition to the fact that $e^{i\pi} + 1 = 0$ and $V - E + F = 2$, Mackenzie includes the identities $\zeta(x) = \Sigma\ 1/n^x = \Pi\ (1 - p^{-x})^{-1}$ and $\zeta(2) = \Sigma\ 1/n^2 = \pi^2/6$, in which the infinite sums are to extend over all positive integers $n \geq 1$, and the infinite product over all primes $p > 1$. Why, one wonders, did he omit Euler's equations of motion for perfect fluids and rigid bodies? Euler, incidentally, never actually wrote $e^{i\pi} + 1 = 0$. The closest he came was $\exp(ix) = \cos x + i \sin x$.

In his chapter on Maxwell's equations, Stewart traces their development from Faraday's notion of a field and deduces from them (in an endnote) that both the electric and magnetic fields satisfy the generalized (three-dimensional) wave equation. He then recounts the detection—in 1886 by Heinrich Hertz—of the waves predicted by Maxwell, their transmission a decade later over a distance of 16 kilometers by Guglielmo Marconi, and the revolution in entertainment and communication made possible by those achievements. Crease, in contrast, focuses on the more complicated form of the equations in the presence of electric charge distributions, and their development by Oliver Heaviside into a tool useful in the design of electrical circuits, while Mackenzie summarizes the evidence indicating that light is an electromagnetic wave.

In his chapter on the Navier–Stokes equations—omitted by both Crease and Mackenzie—Stewart focuses on recent developments in computational fluid dynamics and their relevance to modern automobile and aircraft design. A variety of questions either difficult or impossible to answer by wind-tunnel experiment are approachable via CFD. To emphasize the point, Stewart depicts the computed airflow around a Formula 1 race car, information almost impossible to obtain from wind-tunnel experiments. He also points out that CFD is directly applicable to climate studies, concerning which he provides a good deal of up-to-date information.

Only Mackenzie devotes a chapter to S.S. Chern's generalization $\int_M Pf(\Omega) = (2\pi)^n\ \chi(M)$ of the classic Gauss–Bonnet formula, in which M represents an arbitrary manifold of even dimension $2n$, Ω its Gaussian curvature, Pf the Pfaffian operator, and $\chi(M) = 2 - 2g$ the Euler characteristic of M. Because the "connectivity" g of M represents the number of holes in space enclosed by M, it can assume any of the values 0, 1, 2, . . . , meaning that $\chi(M)$ can assume any of the values 2, 0, –2, –4, ...

After being extricated from Japanese-held China in 1943, Chern (considered then and now to be Cartan's most gifted student) was invited to visit the Institute for Advanced Study in Princeton. While there, he learned that André Weil and Carl Allendoerfer had recently proved a version of the classic ($n = 1$) Gauss–Bonnet theorem that worked for any even-dimensional manifold M. Finding their proof ugly and unenlightening, Chern produced one of his own (published in a six-page paper in 1946) that was neither, and that—by introducing the concept of a fiber bundle—revitalized 20th-century geometry.

It came to light later that fiber bundles are closely related to quantum fields. "To understand the shape of a space," Mackenzie writes, "you need to know what kinds of fiber bundles—or, essentially the same thing, what kinds of quantum fields—can be erected on that space." As he adds later, "Michael Atiyah and Isidore Singer made the links between math and physics even more explicit" by giving "a proof of the Chern–Gauss–Bonnet theorem (and quite a bit more) that proceeds directly from solutions of the Dirac equation!"

Well written and entertaining, all three books are also replete with interesting details that add to what most mathematicians already know about the equations described and the events chronicled. Generally, Crease seems the most philosophically inclined of the three authors, Stewart has the most to say about applications, and Mackenzie delves most deeply into the historical development of purely mathematical ideas. All three books can be recommended to any reader with even a casual interest in mathematics and/or its history.

James Case writes from Baltimore, Maryland.