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Lorenz System Offers Manifold Possibilities for Art

By Barry A. Cipra

The Lorenz attractor has been a favorite of mathematical lepidopterists ever since chaos theory took off in the 1970s. First described by MIT meteorologist Edward Lorenz in 1963, the delicate butterfly wings that unfurl from an unassuming cocoon of simple equations have graced the pages of innumerable papers in dynamical systems. But for Bernd Krauskopf and Hinke Osinga, there's an equally attractive geometric counterpart to Lorenz's eponymous point set: a smoothly convoluted surface known as the Lorenz *manifold*.

Krauskopf and Osinga, longstanding collaborators in the Department of Engineering Mathematics at the University of Bristol in the U.K., presented recent results of their studies of the Lorenz manifold in a minisymposium devoted to the legacy of Edward Lorenz at SIAM's dynamical systems conference in Snowbird last May. Their work has focused in part on visualization of the complicated surface, and one of the results of those efforts is literally tangible. In a collaboration with Krauskopf and Osinga, sculptor Benjamin Storch created a precise realization in stainless steel of a portion of the Lorenz manifold.

The Lorenz system is given by a set of three ordinary differential equations or, more precisely, by a parameterized family of ODEs:

$$x' = \sigma(y - x)$$
$$y' = \rho x - y - xz$$
$$z' = xy - \beta z,$$

where the parameters σ, β, and ρ are positive real numbers. The classic Lorenz equations—and the ones used for the sculpture—use the values $\sigma = 10$, $\beta = 8/3$, and $\rho = 28$. This endows the system with fixed points at (0,0,0), $(6\sqrt{2}, 6\sqrt{2}, 27)$, and $(-6\sqrt{2}, -6\sqrt{2}, 27)$. The latter two are, roughly speaking, the eyes of the butterfly wings.

According to legend, Lorenz devised these equations as a toy weather model (picking parameter values that would produce results mimicking unstable convection patterns in the atmosphere) and had a computer (a Royal McBee LGP-30) churn out numerical solutions. At one point he restarted the computation using intermediate values from the computation's output, only to discover that seemingly insignificant roundoff—the machine computed to six digits, but reported only three—caused the restarted computation to quickly diverge from its previous output. Lorenz's report of this sensitivity to initial conditions was one of the slow-burning embers that ultimately erupted in the blaze of chaos theory that swept across physics in the 1980s.

Though emblematic of chaos, the Lorenz system was not truly known to be chaotic until 2002, when Warwick Tucker, now at the University of Uppsala in Sweden, proved that the attractor is indeed "strange," the mathematical term of art for an attractor that displays sensitivity to initial conditions. (Actually, even "attractor" is a term of art. There is a subtle but significant difference between an attractor and a mere attracting set. The rigorous study of dynamical systems can be a definitional headache.)

Cryptically speaking, the Lorenz attractor is what you see when you watch particles swept along by the equations' dynamics; the Lorenz manifold is what you *don't* see. Less cryptically, the manifold consists of trajectories that are sucked toward the fixed point (0,0,0). The linearized system there has two negative eigenvalues, which means that these trajectories constitute a smooth two-dimensional surface.

Two trajectories are easy to spot precisely: the positive and negative z-axis. But everything else on the Lorenz manifold can only be computed numerically; there are no explicit formulas that describe the surface. Krauskopf and Osinga have developed methods for doing the requisite computations. They have also proved theorems that guarantee the accuracy of the resulting approximations.

Their basic idea for computing the Lorenz manifold is to start with a small circle around the origin, in the plane of the manifold, and then "grow" the circle outward in a radially uniform fashion (i.e., in some sense ignoring the dynamical equations). The result at each stage is a closed curve numerically approximated by a polygon. The next closed curve is produced by the dynamics in a rather subtle way, through suitable boundary-value problems that specify the vertices of the new polygon. When computed correctly, each new closed curve (an approximate geodesic level set) consists of points a fixed (geodesic) distance from those of the previous curve. The polygonal approximation of the new closed curve is refined, as needed, by increasing the number of vertices.

It might seem to be "truer" to the dynamics to simply transport a closed curve by following the backward flow for an instant (say Δt) to obtain a new closed curve. But any arbitrarily chosen initial small circle very quickly deforms badly—and thus fails to give a useful representation of the surface. The geometric approach by Krauskopf and Osinga deals with this problem and generates a nice mesh representation of the surface.

Figure 1. *"Manifold," a stainless steel sculpture by Benjamin Storch. Created in collaboration with Bernd Krauskopf and Hinke Osinga, the sculpture hints at the fascinating dynamics implicit in the equations of the Lorenz system.*

As the computed closed curves expand away from the initial circle, they begin to twist and turn like a smoke ring caught in a gentle—and symmetric—vortex. (The manifold respects the equations' 180-degree rotational symmetry about the z-axis.) In essence, the Lorenz manifold organizes the trajectories that pass near the origin as they spiral toward the butterfly wings of the attractor, flitting from one to the other.

For the sculpture, Krauskopf and Osinga computed the Lorenz manifold to a geodesic distance of 140.75. An 8-cm-wide ribbon of stainless steel, polished on one side and brushed on the other, the sculpture shows the outermost 20-unit-wide band of the manifold.

Storch fashioned the sculpture, which is titled "Manifold," by welding together three pairs of laser-cut sheet metal pieces (see Figure 1). The pieces in each pair are identical, in accord with the rotational symmetry of the manifold. The bottom two barely bend out of the plane spanned by the two stable eigenvectors (and were cut from a thicker sheet of steel to help support the overall weight of the sculpture), but the others required careful hammering of the flat metal to produce the manifold's graceful curvature.

Comparison of photographs of the sculpture with computer-generated images, Krauskopf and Osinga point out, reveals the extent of Storch's skill: The surfaces match up almost perfectly. Indeed, the mathematicians enjoy showing a short video of the manifold slowly turning, then polling viewers on whether they are watching an actual video or a computer animation. (For the record, this reporter guessed right.)

The stainless steel piece is not the researchers' first foray into a physical rendering of the Lorenz manifold. In 2003, Osinga created a crocheted model of the entire surface out to geodesic distance 110.75. Osinga's crocheted piece (Figure 2) adheres closely to the numerical algorithm for computing the manifold: Where the algorithm adds a mesh point, the crochet instructions add a stitch. Attached to a rod representing the z-axis and stiffened with garden wire surrounding its rim and two special trajectories to the origin, the 25,511-stitch piece gives a sense of the Lorenz manifold's complex shape.

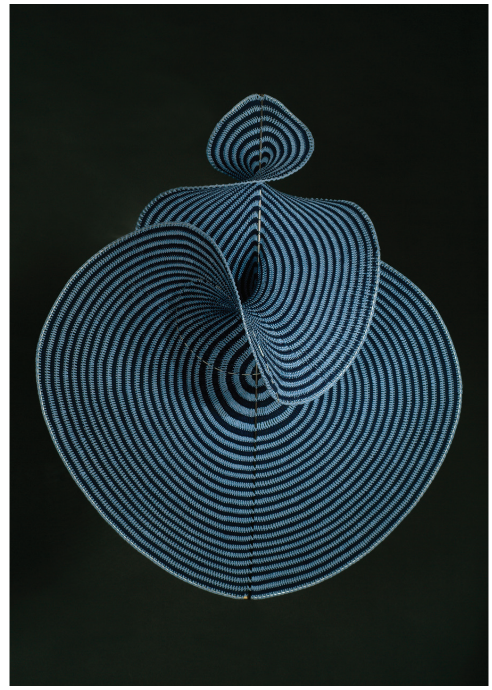

Figure 2. *Model of the Lorenz manifold crocheted by Osinga (2003).*

In their more purely mathematical work, Krauskopf and Osinga have been studying bifurcations in the Lorenz system as the ρ-parameter varies (keeping σ and β fixed at 10 and 8/3). The Lorenz manifold and its one-dimensional counterpart, the unstable manifold for the origin (i.e., the trajectory that heads to the origin if you reverse time), interact in complicated ways with the stable and unstable manifolds associated with the other fixed points and certain periodic orbits of the system. In particular, the Lorenz manifold and the unstable manifolds of the equations' other two fixed points (which are both two-dimensional surfaces) intersect as an infinite collection of "heteroclinic orbits."

A typical heteroclinic orbit, once it leaves the vicinity of its initial fixed point, darts back and forth, loosely looping along the Lorenz attractor, until it finally heads toward the origin. The darting and looping can be described symbolically as a list of r's and l's, each symbol specifying a loop around the "right" or "left" fixed point (see Figure 3). As described in a paper with Eusebius Doedel of Concordia University in Montreal, Krauskopf and Osinga have computed these orbits with up to nine loops, a total of 512 cases, and investigated their dependence on ρ.

In brief, the heteroclinic orbits all stem from a bifurcation known as a "homoclinic explosion point" at ρ ≈ 13.9265. What's more, each heteroclinic orbit persists as the ρ-parameter increases, up to a "fold," beyond which the continuation doubles back in ρ, with the orbit ending at another homoclinic explosion point. The researchers have mapped out the combinatorial structure that determines which heteroclinic orbits end at which homoclinic explosions.

Much remains to be learned about the Lorenz system, such as how the Lorenz manifold responds to changes in the other two parameters, σ and β. Almost half a century after the Royal McBee performed its tantalizing computations, the Lorenz attractor and its associated dynamics continue to attract mathematicians, like moths to a flame.

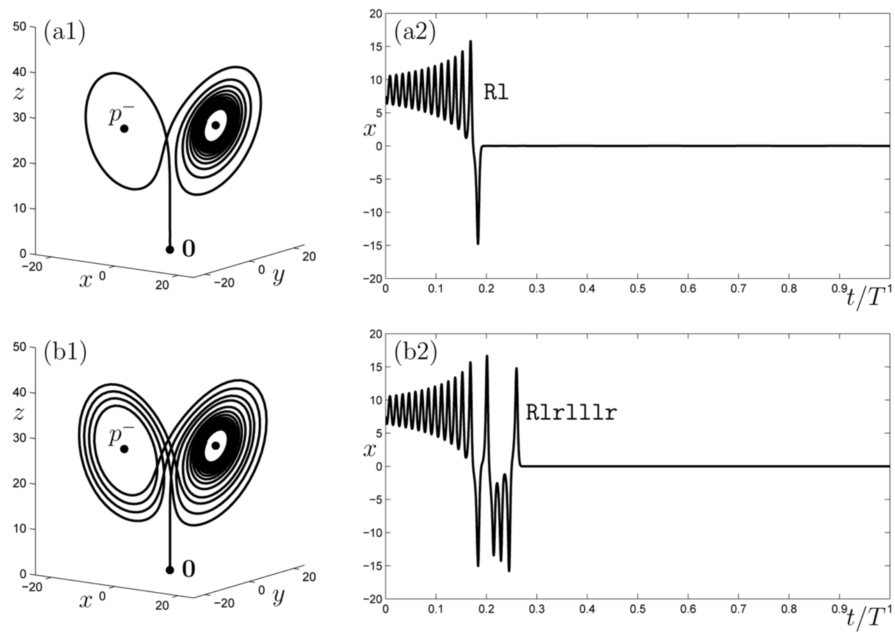

Figure 3. *Like soldiers obeying a sadistic drill sergeant, heteroclinic orbits connecting the origin to the Lorenz system's other two fixed points lurch from side to side after emerging from a spiral. Their forced march can be described by a finite sequence of r's and l's ("rights" and "lefts"). In the two examples shown here, the orbits begin with a spiral, symbolically denoted "R," around the fixed point with positive x-value. Reprinted with permission from* Nonlinearity, *Vol. 19, No. 12.*

For Further Reading

■ B. Krauskopf, H.M. Osinga, and B. Storch, *The sculpture Manifold: A band from a surface, a surface from a band*, in Reza Sarhangi and Carlo Séquin, eds., Proceedings of Bridges Leeuwarden: Mathematical Connections in Art, Music, and Science, Leeuwarden, 2008, 9–14.

■ H.M. Osinga and B. Krauskopf, *Crocheting the Lorenz manifold*, The Mathematical Intelligencer, 26:4 (2004), 25–37.

■ E.J. Doedel, B. Krauskopf, and H.M. Osinga, *Global bifurcations of the Lorenz manifold*, Nonlinearity, 19:12 (2006), 2947–2972; with multimedia supplement at http://www.iop.org/EJ/mmedia/0951-7715/19/12/013/.

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