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Math from the Pan: Fantastic Voyage Through a French Fry

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The food industry offers a surprisingly rich variety of interesting problems to applied mathematicians: coffee brewing, chocolate manufacturing, mechanical and chemical processing (mixing, leavening, extruding) of dough, and the drying, freezing, boiling, baking, roasting, emulsifying, bottling, and wrapping of a wide variety of foods. The list is impressively long, and the fact that enormous amounts of money move all these processes along is not a minor detail.

Mathematicians have made valuable contributions to the modeling and optimisation of some of these processes. Among them is one of the most challenging areas in the food industry: frying, which combines mechanics, thermodynamics, and chemistry, along with medicine and dietology. Food engineers have studied frying rather deeply because of its economic importance, but mathematicians have looked into the subject only very recently and, so far, with a limited target: the frying of not-too-thin bodies with sufficiently compact texture by immersion in hot edible oils.

The constraint on thickness is motivated by the wish to exclude mechanical deformation (e.g., the warping that occurs in potato chips); the requirement for “solid” materials precludes the swelling and large-bubble formation that occur inside samples in, for example, batter frying. French fries are a good example of the product we consider. Even in this reduced framework, a mathematical model has to deal with many coupled processes, including heat transfer from the oil to the sample, vapor production, migration of vapor out of the sample, and crust formation (see Figure 1). In this article we illustrate the main ideas of a model that, we believe, has given convincing answers to some of the key questions about French fries. A detailed description of the model can be found in [2].

For lack of space, we omit here the analysis of crust formation, despite its importance in determining taste. We also ignore the chemistry, which means that our viewpoint is very partial. Nevertheless, a correct mathematical description of the thermodynamical aspects is a necessary step in the development of new techniques, which for French fries include vacuum frying: Next-generation French fries will be vacuum-fried, i.e., cooked at low pressure and low temperature, eliminating most potential noxious products [3].

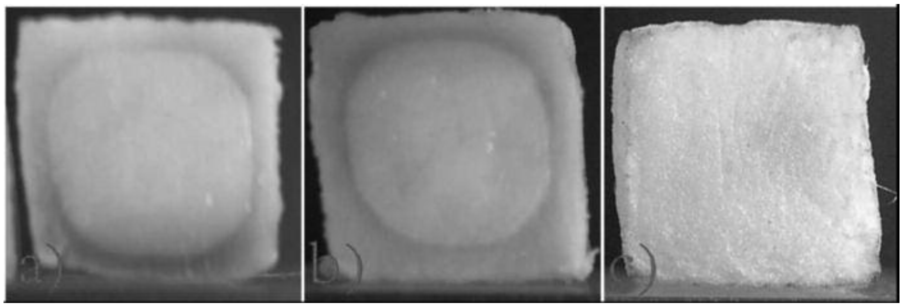

Figure 1. *Potato samples in cross section at different stages of frying (a and b), emphasizing the presence of an interface, followed by the formation of a crust (c). These samples are from the kitchen of Luis T. Villa in Salta, Argentina, who first drew the authors’ attention to the problem some years ago.*

Driving the Vapor Out

The formulation we present is one-dimensional, meaning that the “potato slice” being modeled is sufficiently thick. A nonmathematical reader would immediately object: “Applied mathematicians have an inveterate tendency to cheat! First they talk about French fries, then they serve some tasteless theoretical sample, and of course on an infinite dish!” Touché! But the criticism is only half true. The few experiments described in the literature [1] refer to large samples of the type we consider (1.25 cm thick, 7 cm in diameter), and with good reason: Sufficient space is needed for treating thermocouples.

Writing down the corresponding multidimensional formulation is not difficult (the mathematics, of course, is much more difficult), but the first step is to acquire confidence in the modeling by adopting a more schematic approach. The real “cheating” lies elsewhere: If frying a thick potato slice has little to do with the industrial process, simulating a French fry with a single slender body is equally far from the real thing.

Indeed, whether in an industrial plant or in a home kitchen, preparing French fries involves the sudden immersion of a large quantity of cut potatoes in hot oil. The resulting temperature difference is considerable, particularly with frozen French fries (in which case the surface has a coating of ice). This gives rise to a deep alteration of the thermal conditions of the system. The oil temperature drops in a nonuniform way, inducing strong convective motion, which is soon enhanced by massive production of water vapor bubbles. Modeling all these events is not impossible but can be attempted only once the basic process of frying a single sample in an oil bath at constant temperature has been understood with sufficient clarity.

Thus, we consider a slab $-L < x < L$ of a porous material saturated with water. At time $t = 0$ the temperature T is uniform and below the boiling point of water at atmospheric pressure. The plane $x = 0$ is a surface with no heat or mass exchange; therefore, we set our model in $0 < x < L$ with the boundary condition

$$k_{sat}\frac{\partial T}{\partial x}\Big|_{x=L} = \chi\left(T_{oil} - T\right)\Big|_{x=L}, \quad (1.1)$$

where k_{sat} is the heat conductivity of the saturated medium, T_{oil} is the temperature of the oil, which is greater than T_0, the boiling temperature of water at atmospheric pressure, p_0. The coefficient χ is assumed to be constant during the preheating stage, i.e., as long as $T(x, t) < T_0$. Obviously, the preheating stage is standard: The governing equation is the heat conduction equation with the given initial temperature, no flux at $x = 0$, and condition (1.1) at $x = L$.

At time t_0, at which $T(L, t_0) = T_0$, the situation suddenly becomes complicated, because the heat transfer rate at $x = L$ can be modified by the onset of vaporization. This early stage of the process has to be discussed separately. A later stage, also affecting the rate of heat exchange, is the formation of the crust. For the moment, we concentrate on how vaporization develops within the sample. From this perspective, frying means simply: (i) heat propagation, (ii) vapor production, and (iii) migration of vapor toward $x = L$, followed by its expulsion into the oil.

The main question that arises at this point is: How is vapor driven out of the sample? As the transport mechanism of vapor we use Darcy's law:

$$q_v = -K_v \rho_v \frac{\partial p}{\partial x}, \qquad (1.2)$$

where q_v is the vapor mass flux, K_v the hydraulic conductivity for the vapor, p the pressure, and $\rho_v = \rho_v(p, T)$ the vapor density in the pores. But how is the pressure gradient created? To answer, we look to the liquid–vapor interface $x = s(t)$, where we impose the following conditions: (i) mass balance, (ii) enthalpy balance, (iii) continuity of temperature, and (iv) water–vapor thermodynamical equilibrium. The latter condition is expressed by Clapeyron's law:

$$\begin{aligned} &p = p_0 \exp\left[\frac{\lambda}{R}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right], \\ &x = s(t), \qquad\qquad (1.3) \end{aligned}$$

with T measured in degrees Kelvin; λ is the latent heat of vaporization, and R the universal gas constant. Here T is not fixed on the interface; rather, as discussed later, T increases with time (albeit very slowly), producing, thanks to the exponential behavior of (1.3), a significant increase in pressure and, thus, the required pressure gradient.

General Setting of the Model

Before analyzing the interface conditions, we describe the phenomena occurring in $0 < x < s(t)$ (saturated region) and in $s(t) < x < L$ (vapor region). The unknowns are temperature and (for the vapor region only) pressure. In the saturated region we consider the heat conduction equation with the initial condition attained in the preheating stage and no heat flux at $x = 0$.

In the vapor region Darcy's law (1.2), where $K_v = K_v(T)$, $K_v'(T) > 0$, allows us to write down the mass balance of vapor and the heat conduction–advection equation. Because of space constraints, we state here simply that they constitute a nonlinear parabolic system in the unknowns p, T.

The conditions at the interface were given earlier; in particular, (1.3) links pressure and temperature. We can now mention that mass and enthalpy balance can be derived as the Rankine–Hugoniot conditions for the corresponding equations in the bulk of the material. Doing so, we arrive at the equations (2.1) and (2.2) (denoting the right and left limits, respectively):

$$\varepsilon(\rho_w - \rho_v)\dot{s} = \rho_v K_v \frac{\partial p}{\partial x}, \qquad (2.1)$$

$$\begin{aligned} &\varepsilon\rho_w\lambda\dot{s} = -(1-\varepsilon)k_s \\ &\left[\frac{\partial T}{\partial x}\right]^+ + k_{sat}\left[\frac{\partial T}{\partial x}\right]^- . \qquad (2.2) \end{aligned}$$

Here, ρ_w is the water density; ρ_v can obviously be neglected in (2.1).

Playing with Time Scales

A fundamental step in the study of the model just sketched is the introduction of nondimensional variables, which has to be done in a "smart" way. Several time scales can be defined: t_d, t_s represent typical heat diffusion times in the saturated region and in the solid matrix, respectively; t_λ is the typical time scale of the vaporization process (involving λ) and t_v the typical time scale of the vapor transport process. Be-cause the main phenomenon is the production of vapor, the natural time scale is t_λ. After rescaling the mass balance equation, we see terms that are multiplied by the ratio t_λ/t_v, which turns out to be large. We can thus simplify the equation by requiring that the sum of such terms be zero. The resulting equation can be interpreted as $\partial/\partial x(\rho_v K_v(\partial p/\partial x)) = 0$, in nondimensional variables.

This leads us to conclude that the vapor flux is reasonably approximated by a function $f(t)$ only of time, so far unknown. As a consequence, the interface condition (2.1) reduces to

$$\begin{aligned} &-f(t) = \frac{1}{\Theta_\rho}\dot{s}, \\ &\qquad \Theta_\rho = \frac{t_\lambda}{t_v}\frac{\rho_v^0}{\rho_w} \simeq 0.1. \qquad (3.1) \end{aligned}$$

In the new formulation, pressure becomes a functional of temperature. The numerical simulations presented below refer to this limiting case.

The Onset of Vaporization

Now that we have a model, mathematicians will want to see a proof of well-posedness. Engineers, for their part, will expect to see simulations and comparison with experimental data.

With apologies to mathematicians: Even in its simplified version, the problem is too complicated for treatment with the usual tools. The main difficulty is that equation (3.1) leads to a nonlocal free boundary condition of a particularly unfriendly type. We leave related theoretical investigations as an open problem. But stay tuned! Those who love free boundary problems will find some surprises.

Indeed, a priori knowledge of the initial development of the interface is extremely useful for the numerical computation. For this reason, we performed a qualitative study of the onset of vaporization. To understand the unexpected peculiarities we encountered, consider the following ideal case: A heat-conducting material, occupying the half space $x > 0$, is heated by an external source up to its melting point. Typically, we would refer to a boundary condition of the type (1.1), which we now rewrite in the nondimensional form:

$$-u_x = h(\hat{u} - u), \quad x = 0, \qquad (4.1)$$

where u denotes temperature ($u = 0$ is the melting point, $\hat{u} > 0$ the external temperature, and h the heat exchange coefficient). If h does not change with a phase change, we have a standard Stefan problem. If, by contrast, the phase change alters the structure of the surface in such a way that h decreases, we are in trouble. This will happen if, for instance, the surface of the solid is "black" while the surface of the liquid is "white" (i.e., less capable of absorbing heat). While such an example may appear simplistic, in our case it makes a lot of sense: Prior to vaporization, the surface exchanging heat with the oil is a mixture of the solid matrix and water; when vaporization starts, the latter is replaced by the outgoing vapor.

Coming back to our ideal example, the heat input rate should be increased at the onset of phase change to provide the extra amount needed to balance the latent heat. If the incoming thermal flux is reduced, it becomes impossible to sustain the phase change. But if we do not allow the phase change, the temperature will rise above the melting point. How are we to work around such conflicting conditions?

Simply by waiting! We let the boundary temperature stay at $u = 0$ and read equation (4.1), i.e., $-u_x = h\hat{u}$, as the evolution law of h. Melting will start only when h has reached the value corresponding to complete "melting of the surface" (so to speak). In other words, it is necessary to let the surface go from black to white, through all levels of gray, in the course of some waiting time, which is easily computable. In [1] we examined the behaviour of the free boundary in both cases: h unchanging or decreasing on vaporization.

If the heat transfer coefficient remains unchanged, the thickness of the fully vaporized region increases as $t^{3/2}$ while pressure on the interface increases as t^2, and we can provide the exact coefficients. If, however, the heat transfer coefficient decreases, then we have a waiting time and these quantities become t^2 and $t^{5/2}$, respectively.

The waiting time phenomenon is certainly very interesting in general, but for the specific case of frying, its duration is very short in comparison to the main time scale t_λ of vaporization. Therefore, we decided to perform simulations with no change in the coefficient χ in (1.1). The theoretical analysis of the early stage of vaporization proved to be essential to start the numerical procedure.

Matching Experimental Data

We performed numerical simulations with reference to the sample considered in [1] (thickness, 1.25 cm; initial temperature, 298 K; temperature of the oil, 463 K; for brevity, we omit the list of other parameters). Figures 2(a)–(d) show the temperature profiles at four times after the onset of vaporization, clearly in satisfactory agreement with the measurements indicated by the thermocouples.

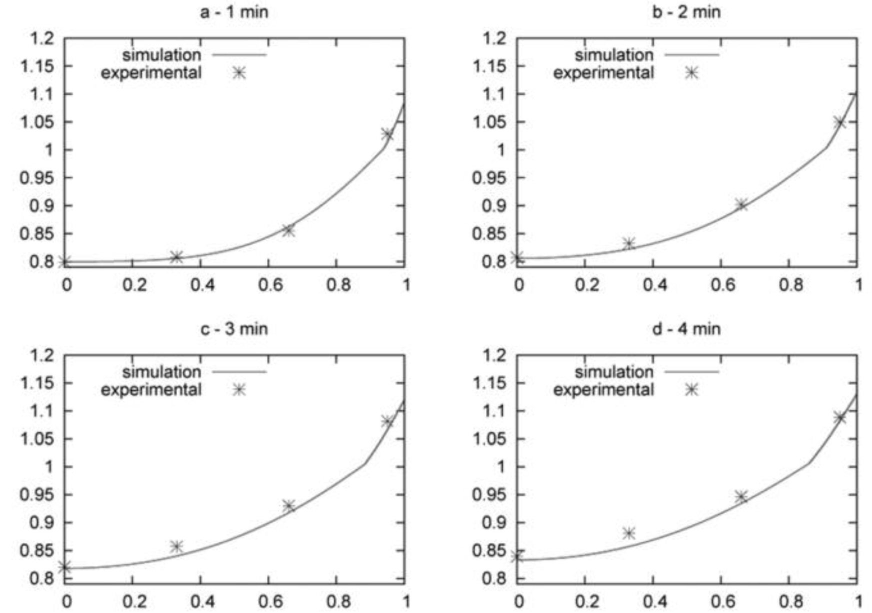

Figure 2. *Experimental vs. computed temperature profiles.*

In conclusion, we believe that French fries have been sufficiently mathematized and we hope that readers who have read this far have enjoyed the voyage! Does the insight gained enhance the taste of French fries? Not at the moment, but the voyage is not finished and there is a lot of fun ahead for applied mathematicians!

References

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