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Dear Matrix Maven

Matrix Mathematics: Theory, Facts, and Formulas with Application to Linear Systems Theory. *By Dennis S. Bernstein, Princeton University Press, Princeton, New Jersey, 2005, 726 pages, $89.50.*

BOOK REVIEW

By Philip J. Davis

I have a soft spot for dictionaries, encyclopedias, compendia, formularies, and the like. In my own work, I have found compendia useful in research, in laying out course material, and as a source of problems; occasionally, they have suggested new possibilities. I even keep a few compendia near my bed and rely on them to fall asleep when I lack more sensational material.

Years ago, while employed at the National Bureau of Standards, I contributed two chapters to the well-known, much used and much cited *Handbook of Mathematical Functions*, a.k.a. Abramowitz and Stegun, a.k.a. A&S. The earlier "Bateman Project," a compendium of special functions (A. Erdelyi et al., *Higher Transcendental Functions*, 1953–54), was much used by the contributors to A&S, and by many others. Very likely my experience with these works intensified my soft spot.

Bernstein's *Matrix Mathematics* is a compilation of definitions, theorems, propositions, and facts, facts, facts galore. There are identities and there are inequalities. Here and there, there are proofs or hints for proofs. Bernstein's bibliography is extensive, and his indices and glossaries are well thought out and useful.

A list of the chapter titles will give readers an idea of the book's coverage: Preliminaries, Basic Matrix Properties, Matrix Classes and Transformations, Matrix Polynomials and Rational Transfer Functions, Matrix Decompos-itions, Generalized Inverses, Kronecker and Schur Algebra, Positive-Semi Definite Matrices, Norms, Functions of Matrices and their Derivatives, The Matrix Exponential and Stability Theory, Linear Systems and Control Theory.

To give the flavor of the text, here are a few examples that I wasn't aware of previously, culled from the enormous set of results in the book:

■ Fact 5.14.23. If A is a matrix with complex elements, then $\det(A)$ is real if and only if A is the product of four hermitian matrices. And four is the smallest number that will work generally.

■ Proposition 8.2.4 provides us with an explicit formula (too complicated to be reproduced here) for the inverse of a positive definite matrix partitioned into four square submatrices and given in terms of the inverses of the partitions.

■ Fact 10.8.13. $(d/ds)\det(A + sB) = \mathrm{tr}[B(A + sB)^A]$. tr designates the trace, and the superscript A designates the adjugate matrix.

Linearity as a broad mathematical and applied mathematical subject, with all its theories, facts, and formulas, is now far beyond what one person can know or care for. It is not surprising that the author, a distinguished professor of aerospace engineering at the University of Michigan, has shaped his compendium toward linear systems theory. Even with this limitation, the book runs to 560 pages plus ancillary material, 70 pages of which are dedicated to linear systems and control theory. But the coverage will clearly be of use not only to systems theoreticians, but to many others as well.

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Do not underestimate the labor of love—even today—involved in making compilations: deciding who your target readership is, scanning hundreds, perhaps thousands of documents, selecting what might conceivably be useful, arranging the material in some kind of sensible order, unifying notations, eliminating typos, and so forth. This is not to mention arguing with your publisher, who may have set limits on the number of pages that is financially feasible. I acknowledge Bernstein's labors and take my hat off to him.

To appreciate better the nature of the task, consider that my search engine came up with 192,000 hits for the term "sparse matrix" (which is not one of the topics covered in the book) and 19,000 hits for "matrix exponential" (which is). The sad fact is that compilations—no matter how focused, no matter how extensive—are always incomplete, and always call for updates. Incompleteness does have an upside: It is a happy indication that the field is still alive and producing new and interesting stuff daily.

A bit of the history of mathematics is of interest here. In the historic order of things, determinants preceded matrices by a good 150 years. Rudimentary forms of determinants are found in Leibnitz, and even much earlier. Moreover, many of the facts now displayed under the rubric of matrix theory were discovered and developed within the context of determinant theory. Then, beginning with Sylvester and Cayley in the mid-1800s, determinants were very slowly taken over by matrix theory. In the process, determinants were shoved to the side as poor offspring.

Later on, suffering the same fate as determinants, matrices were shoved to the side conceptually by linear algebra—a designation that seems still to reign supreme as regards undergraduate courses. Who needs explicit bases? Who needs coordinates? And linear algebra spread out to encompass multilinear algebra, tensor theory, and so forth. Yet despite the engorgement of matrix theory—even as one large corporation is swallowed up by a still larger corporation—fresh winds of change were in the air. Matrices were not swallowed up completely. Duncan, Frazer, and Collar were interested in computation, and their *Elementary Matrices and Some Applications to Dynamics and Differential Equations* (1946) was influential at that early date. Then: Ta da! The omnipresence of matrices in applications and the vast possibilities offered by digital computers burst on the scientific world to restore matrices to their well-deserved importance.

The word "matrix" has now gone commercial: a linguistic spin-off. Have you seen the *Matrix* movies? Have you searched for antibiotics on Medical-matrix? Do my English readers realize that Matrix Chambers is ready to help them with their legal problems? I like to think that the popular use of the word started with young people who studied linear algebra in college; deciding upon graduation that there was more money in business or in PR than in math, they nevertheless remembered the word "matrix" and thought it carried the cachet of the "cutting edge" of scientific and technological innovation—and hence of state-of-the-art knowledge in any field whatsoever.

Even determinant theory as such remains alive and kicking. Consider the work of Sir Thomas Muir (1844–1934). Muir spent a lifetime developing and accumulating material related to determinants. He produced more than 320 research papers, most of them on determinants. His magnum opus, *The Theory of Determinants in its Historical Order of Development* (reprinted in four volumes: Dover, 1960), covers the period from Leibnitz to 1920. Muir was working on a sixth volume at the time of his death. The persistence, the dedication, and the sheer quantity of output of Victorian authors in all fields boggle my mind. For a forthcoming issue of *Linear Algebra and its Applications*, titled "Determinants and the Legacy of Sir Thomas Muir," Pieter Maritz has written an extensive discussion of Muir as an educator in South Africa and as a mathematician.

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The publication of compendia targeted to broad readerships is big business. CRC Press turns them out in some twenty disciplines. (See, for example, my review of Eric Weisstein's hardly concise *CRC Concise Encyclopedia of Mathematics*; "Of Making Compilations There Is No End," *SIAM News*, Volume 33, Number 2, March 2000.) Why, generically speaking, are printed compendia still of use? Who needs them, what with the Internet, CD-ROMs, databases, and search engines? But one does need them, and the first reason that pops into my mind is convenience.

I have not seen an in-depth discussion of the limits of search engines, but I'll mention a few difficulties or frustrations that I'm sure my readers have experienced. One such problem is semantic ambiguity. Perhaps you're interested in the construction of spears. So initially and in all innocence, you type in "spears" and run into zillions of references to Britney.

Even within matrix theory, numerous terms have multiple meanings or early meanings, now discarded: conjugate, group, norm, normal, bipartite, aggregate. Changes of terminology as time goes by are inevitable. An example is eigenvalues = characteristic values = latent roots = roots, with the first mentioned now being the favorite. And there is semiotic ambiguity as different notations arise and compete with one another. There are, for instance, notations for functionality that simply weren't around when I was a graduate student.

The second difficulty is taxonomical. People "off the street" have thrown questions at me thinking I'm a matrix maven, which I am not. The questions are often of the type: "I've got such and such a matrix. What is known about it?" or "Is there a simple formula that gives . . .?" With high probability, I scratch my head and send the inquiry to the particular flesh-and-blood matrix mav-en of my acquaintance whose work seems to come closest to the question. But I have to admit that the questions I receive often defy currently available taxonomical names, categories, or keyword identification, leaving me with an inability even to query the online world of compendia.

These various considerations lead me to the following policy: When a matrix question is thrown my way, I will now refer my correspondents both to Bernstein's handbook and to such Web sites as math.nist.gov/MatrixMarket. Perhaps—who knows—my correspondents will hit pay dirt or at least will not go berserk trying to find the right combination of words or links to follow.

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