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Mathematics and Synthetic-Aperture Radar Imaging

By Margaret Cheney

The key technology underlying Synthetic-Aperture Radar (SAR) imaging is mathematics—a wonderfully rich mathematics that includes partial differential equations, scattering theory, microlocal analysis, integral geometry, harmonic analysis, group representation theory, and statistics. Nevertheless, radar imaging remains almost completely unknown in the mathematical community.

Radar technology was developed within the engineering community, for good reasons: In the beginning, many of the challenges had to do with generating and measuring microwaves. Most of those difficulties have now been overcome, however, and the remaining problems are mainly mathematical.

In SAR, an antenna is mounted on a moving "platform," generally a satellite or airplane. The antenna transmits a broad beam of microwave radiation that bounces off the ground and other objects; the same antenna measures the scattered waves. Beautiful images are produced from these measurements, with resolution as small as tens of centimeters even for microwave beams that cover tens of kilometers.

Standard antenna theory has a useful rule of thumb: An antenna that is small relative to the wavelength tends to produce a very broad, diverging beam, whereas an antenna with a large radiating area ("aperture") can produce a much more focused beam. The term "synthetic-aperture" refers to the mathematical synthesis of a large aperture from measurements made with a small antenna moving over a long path.

The invention of SAR is generally credited to Carl Wiley, of the Goodyear Aircraft Corporation, in 1951. During the 1950s, universities and industry cooperated to build the first operational systems. In the 1960s, NASA started to sponsor (unclassified!) work in the area, and the first digital proc-essors were developed. In 1978, the SEASAT–A satellite was launched; its SAR system, although in operation for only 100 days, sent back images so obviously useful that it stimulated a great deal of further work in the area. Since 1981, when the Shuttle Imaging Radar series began, many shuttle expeditions have included SAR missions. In the 1990s, many countries sent up SAR satellites, and space probes carrying SAR systems were sent to other planets. The development of SAR and related radar imaging systems continues unabated today.

The Mathematical Model for SAR

A mathematical model for SAR can be derived from the scalar wave equation

$$\left[\nabla^2 - \frac{\partial_t^2}{c^2(\boldsymbol{x})}\right]\mathcal{E}_s(t,\boldsymbol{x}) = p(t)f_s(\boldsymbol{x}), \tag{1}$$

where the product pf_s is proportional to the effective current density on the antenna, where $\mathcal{E}$ denotes one component of the electric field, and where $c(\boldsymbol{x})$ denotes the speed of wave propagation in the material at position $\boldsymbol{x}$ in space. If the location $\boldsymbol{x}$ is in dry air, for example, then to a good approximation, $c(\boldsymbol{x}) = c_0$, where c_0 denotes the speed of light in vacuum. If $\boldsymbol{x}$ is located under ground, $c(\boldsymbol{x})$ might denote the electromagnetic wave speed in soil. At microwave frequencies, most of the scattering takes place in a thin layer at the surface, so in practice only surface values of c are relevant.

The temporal part p of the source is the time-varying waveform sent to the antenna, whereas the spatial part f_s of the source provides for a model with a realistic antenna aperture. Typically, the $\boldsymbol{x}$-support of f_s is fairly small; most satellite antennas are rectangles, roughly 10 meters × 2 meters.

Because the antenna moves along a path γ, we write the source f_s as $f_s(\boldsymbol{x}) = f(\boldsymbol{x} - \gamma(s))$, where s is a parameter (such as arc length) along the path γ. The electric field $\mathcal{E}_s$ thus depends on the antenna location parameter s.

We measure the field at the same antenna, so the measured data d can be thought of as

$$d(t,s) = \mathcal{E}_s(t, \gamma(s)). \tag{2}$$

The full radar imaging problem is to determine the wave speed $c^2(\boldsymbol{x})$ from knowledge of $d(t,s)$.

Typically, the problem is attacked with the help of a variety of assumptions. In particular, multiple scattering effects are almost always neglected, which allows us to use the known field emanating from the antenna in place of the unknown true electric field impinging on the unknown surface. This assumption eliminates a product of unknowns and thus makes the imaging problem linear, but it is also responsible for artifacts in images (see Figure 1).

Another commonly used approximation, the far-field approximation, is useful for imaging small scenes. This approximation assumes that by the time the wavefront from the antenna reaches objects of interest, it is approximately planar.

Ultimately, the single-scattering and far-field approximations together allow us to approximate the radar data by

$$d(t,s) \propto \int p(t - 2\hat{\gamma}_s \cdot \boldsymbol{x}/c_0)V(\boldsymbol{x})d\boldsymbol{x}, \tag{3}$$

where $V = c_0^{-2} - c^{-2}$ denotes the sought-after perturbation in wave speed and where $\hat{\gamma}$ denotes the unit vector from the antenna to the scene center.

We see from (3) that transmission of a perfect δ-like pulse p gives rise to a Radon transform of the desired wave-speed perturbation V.

Mathematical Challenges

Help is needed from mathematicians in the following areas.

- How can we form images in the presence of complex multiple scattering? Without the single-scattering approximation, the imaging problem is nonlinear, because it involves a product of an unknown scatterer and an unknown wave impinging on that scatterer.
- Radar data depends on two variables, so we can hope to make a two-dimensional image. Often, c is assumed to be non-constant only on a known flat surface; this assumption can lead to distortions in the images, as in Figure 2. In fact, in many cases it is the shape of the surface itself that we wish to discover. This can be done with two widely spaced antennas and a technique called interferometric SAR, which, roughly speaking, is based on the idea of binocular vision. What information about the scattering surface can be found in data from a single antenna, perhaps moving on a non-straight (but still realistic) flight path?

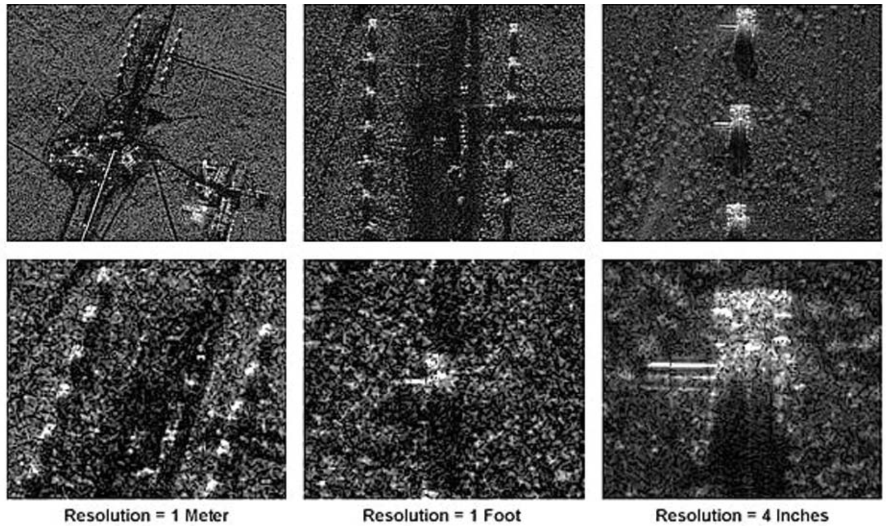

Figure 1. *SAR images of M-47 tanks on Kirtland Air Force Base show the effect of resolution on image interpretability. For each resolution, the lower image is an enlargement (x4) of the upper image. Automatic recognition of objects from SAR images is an open problem. The presence of three gun barrels on the tank at the right is an artifact, caused by neglect of multiple-scattering effects. How can such artifacts be eliminated? Images from Sandia National Laboratories.*

- Forming images of moving objects is problematic: Moving objects can be mispositioned in images or can appear simply as streaks.

- Our current systems collect huge amounts of data, completely overwhelming the available trained image analysts. There is a pressing need for computerized assistance in interpreting SAR images.

- How can we exploit multiple transmitters, perhaps transmitting different waveforms, and multiple receivers? Only small numbers of transmitters and receivers are present, and their locations may not be precisely known or precisely controllable. What waveforms should be transmitted?

The field of radar imaging is wonderfully rich in mathematics and challenging open problems that need mathematical attention!

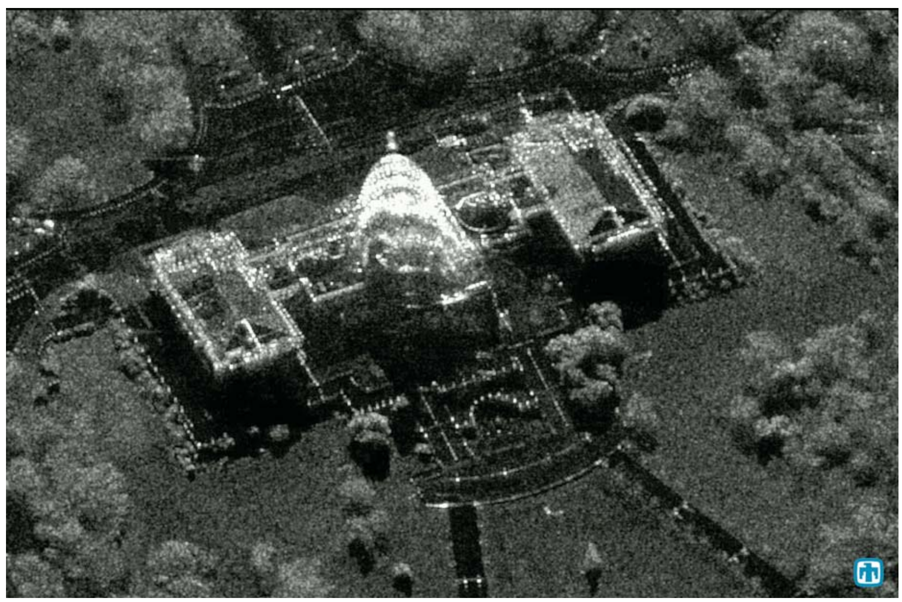

Figure 2. *SAR image of the U.S. Capitol. The dome is distorted, an effect that arises from the assumption that scatterers lie on a flat surface. Interferometric SAR is a two-antenna technique that can be used to find the surface shape. Could the surface be found from single-antenna data? Image from Sandia National Laboratories.*

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