Lax equations and kinetic theory for shock clustering and Burgers turbulence.

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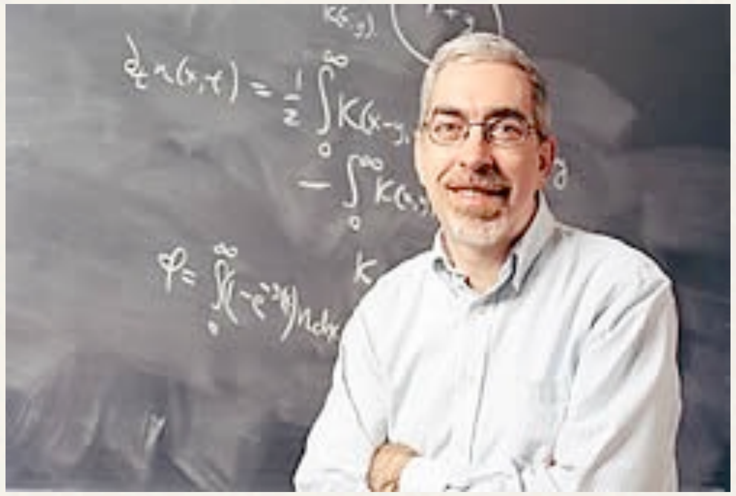
The written version: Menon, Srinivasan, J. Stat. Phys., 2010.

Builds heavily on past work with



Bob Pego,
Carnegie Mellon University

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Jonathan Mattingly (Duke),
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for help on many matters probabilistic, statistical and integrable!

Burgers caricature of turbulence

Solve the differential equation

$$u_t + uu_x = 0, \quad x \in \mathbb{R}, t > 0,$$

with white noise as initial data.

Motivation

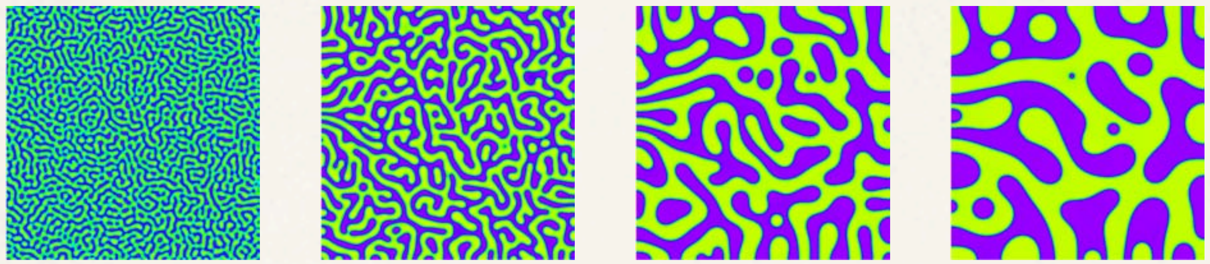
PDE with random initial data or forcing allow us to formulate statistical theories of defects or turbulence.

The emergence of structure from noise is interesting, yet poorly understood, in several basic models.

Vortex coalescence and isotropic 2-D turbulence

2D Navier-Stokes with small viscosity and white noise initial data, Bracco et al.

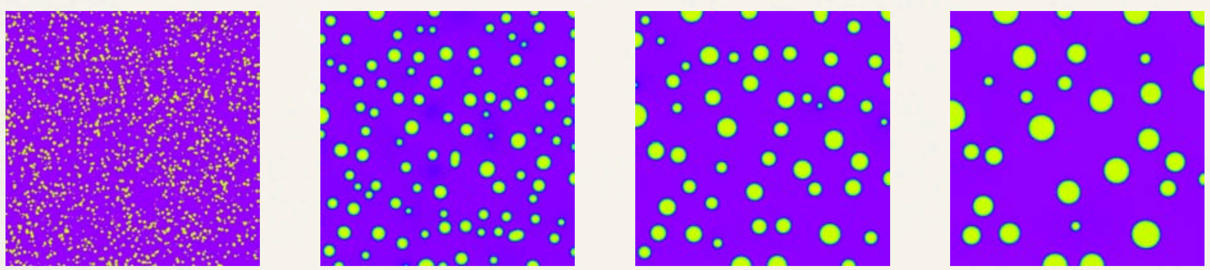
Coarsening in the Cahn-Hilliard equation



(a) $t = 0.025$ (b) $t = 0.125$ (c) $t = 1.25$ (d) $t = 4.875$

FIGURE 5. Evolution with equal mass fractions

Numerics with noise initial data in 2d.



(a) $t = 0.0125$ (b) $t = 0.25$ (c) $t = 2.25$ (d) $t = 6.7$

FIGURE 6. Evolution with set particles

Garcke, Niethammer, Rumpf, Weikard. Arxiv 10.1.1.14.9756

Burgers' motivation for his model

1) The pde is a caricature of the fundamental equations of fluid mechanics.

2) White noise as initial data seems reasonable...

3) Allows us to construct random processes that also solve equations of mechanics, even if in a vastly simplified setting ("Burgulence").

Some basic facts about Burgers equation

1) Global classical solutions do not exist.

2) Weak solutions are not unique.

3) There is a unique entropy solution, which is a vanishing viscosity limit (E. Hopf, 1950).

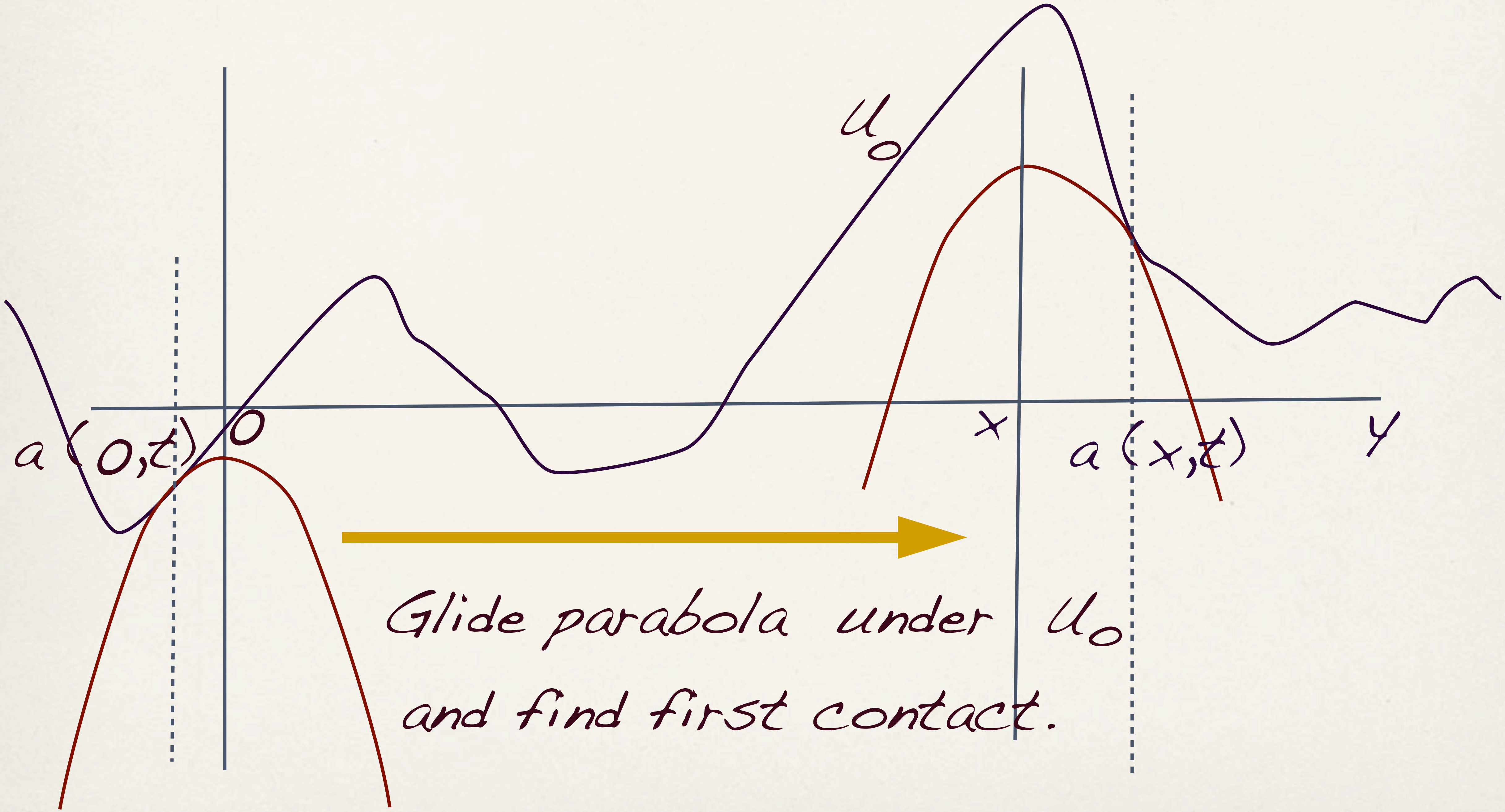
The unique entropy solution, or the Cole-Hopf solution, is given by a variational principle.

$$u(x,t) = \frac{x - a(x,t)}{t}$$

$$a(x,t) = \operatorname{argmin}_y^+ \left\{ U_0(y) + \frac{(x-y)^2}{2t} \right\}$$

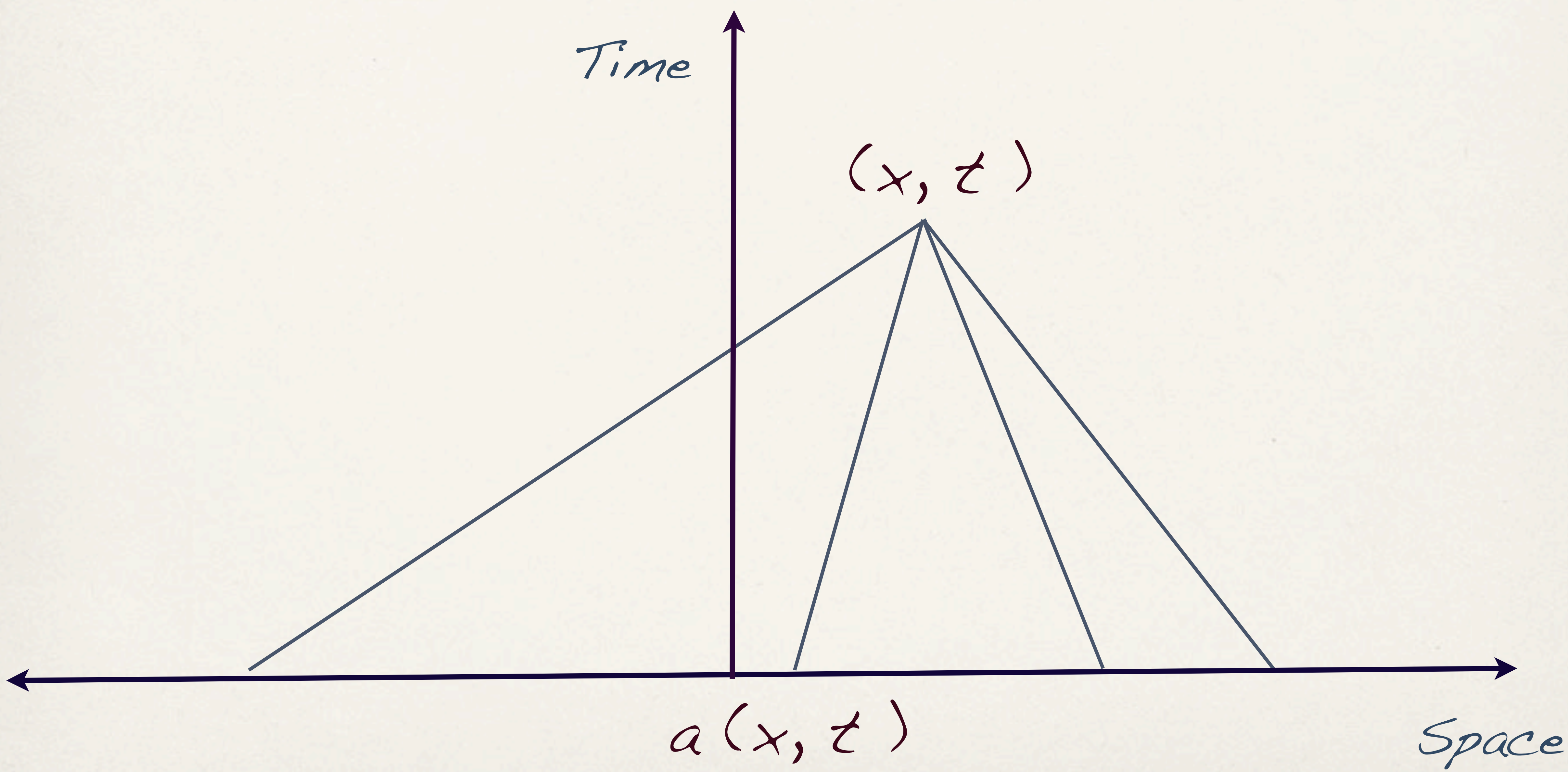
$$U_0(y) = \int_0^y u_0(s)\, ds$$

$u(x,t)$ is the velocity field. U is called the potential and $a(x,t)$ the inverse Lagrangian function. The variational principle is a geometric recipe that uses the potential.

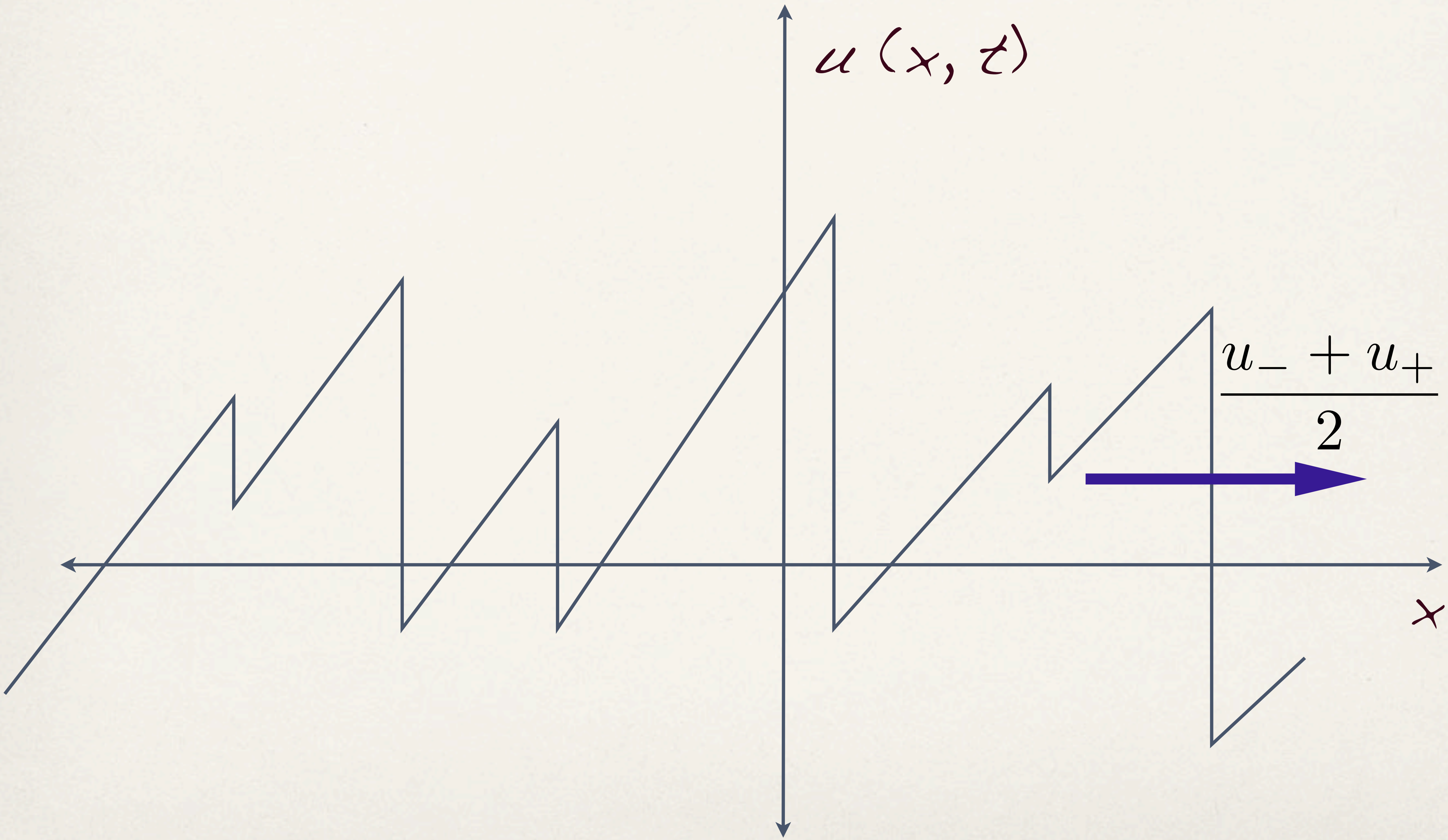


1) $a(x,t)$ is increasing in x.

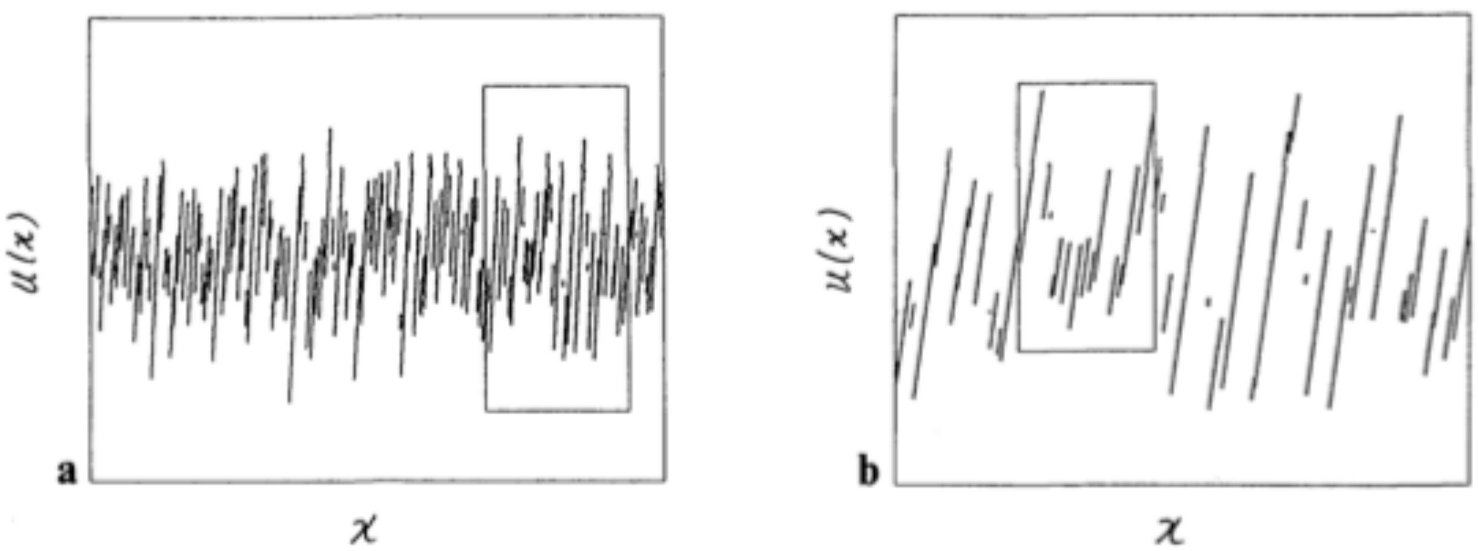
2) $a(x,t)$ gives the 'correct' characteristic through the point (x,t) in space-time.



3) As a consequence, $u(x,t)$ is of bounded variation. Jumps in a give rise to shocks in u. These correspond to 'double-touches' in the geometric principle.



Numerical experiments with white noise data.



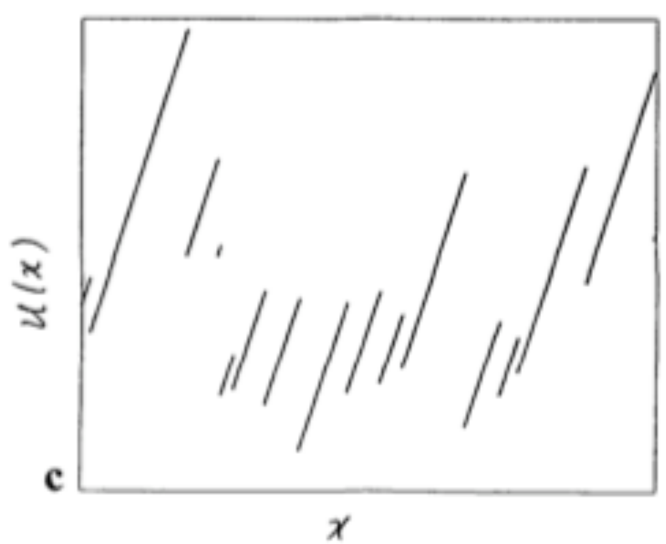


Fig. 4. a Solution $u(x)$, solution to the inviscid Burgers equation at $t = 1$ with white noise initial velocity (type B). **b, c** Show successive small-scale zooming. Note the hierarchy of ramp-like structures with slope 1 and the isolated character of shock points

She, Aurell, Frisch, Commun. Math. Phys. 148, (1992) Burgers worked on this model for a large part of his professional career (1929--1974). He considered various aspects of the problem, including the kinetics of shock clustering in several pioneering articles.

There was a resurgence of activity in the 90's after the numerics of She, Aurell and Frisch. This work was motivated by turbulence.

Interestingly, the same problem also arose in statistics in the 60's, and was solved (in this context) by Groeneboom in the mid- 80s.

A first glimpse at Groeneboom's solution

The one-point distribution of u at time 1 has density

$$p(u) = J(u)J(-u), \quad u \in \mathbb{R}.$$

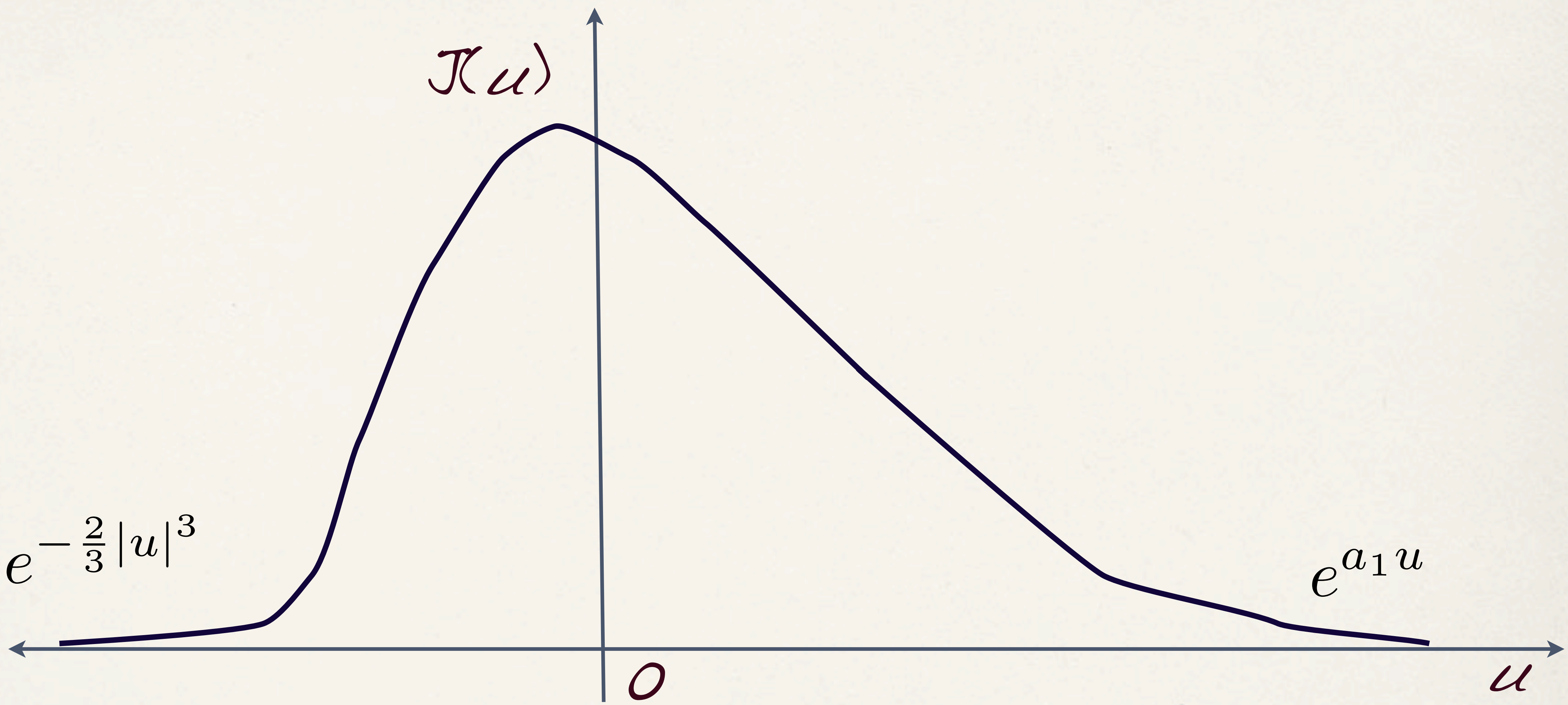
The function J has an explicit Laplace transform

$$\int_{-\infty}^{\infty} e^{-qu} J(u)\, du = \frac{1}{Ai(q)},$$

where $Ai(q)$ is the Airy function.

P. Groeneboom, Brownian motion with a parabolic drift and Airy Functions, Prob.Th. Rel. Fields, 81, (1989).

Classical Tauberian theorems yield asymptotics of J and p



a_1 is the first zero of the Airy function.

$$p(u) = J(u)J(-u) \sim e^{-\frac{2}{3}|u|^3}, \quad |u| \to \infty.$$

The general question

Let f be convex. What can we say about the statistics of the entropy solution to

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbb{R}, t > 0,$$

when the initial data is random?

More precisely, how do we describe the n point statistics for $u(x,t)$ and how does this relate to the coalescence of shocks?

The main results/ conjectures

This problem is exactly solvable for every convex f and suitable random data. Precisely:

1) We derive a kinetic theory of shock clustering for Markov process initial data.

2) The kinetic equations are a completely integrable Hamiltonian system. They describe a principle of least action on 'Markov groups'.

3) The equations are exactly solvable (with blood, sweat and tears) via theta functions.

Some comments

Burgers turbulence is one half of what is often known as "Burgers/KPZ turbulence".

Folklore: "Burgers/KPZ turbulence is solvable".

Our work addresses what this means in terms of mathematical structure and explicit solutions.

For KPZ, there is interesting recent work in roughly the same spirit by Amir, Corwin and Quastel and Sasamoto and Spohn.

Some comments (contd.)

There are close ties with random matrix theory in both problems (exact solutions, integrable structure, combinatorial models), but still a lot that we don't understand.

Burgers equation and Brownian motion are special, but not that special. Arbitrary convex f and the Markov property in space is enough. (There were no previous results for general f).

The general question again, more precisely

Let f be convex and smooth. Determine the statistics of the entropy solution to

$$\partial_t u + \partial_x f(u) = 0, \quad x \in \mathbb{R}, t > 0,$$

when the initial data is a Markov process with only downward jumps.

Why Markov processes in x?

1) if the initial data for Burgers equation is white noise, then the entropy solution is a Markov process in x (Burgers, Groeneboom, Avallaneda-E).

2) key insight of Duchon and coworkers : Burgers equation (formally) preserves the class of Levy processes and possibly other processes.

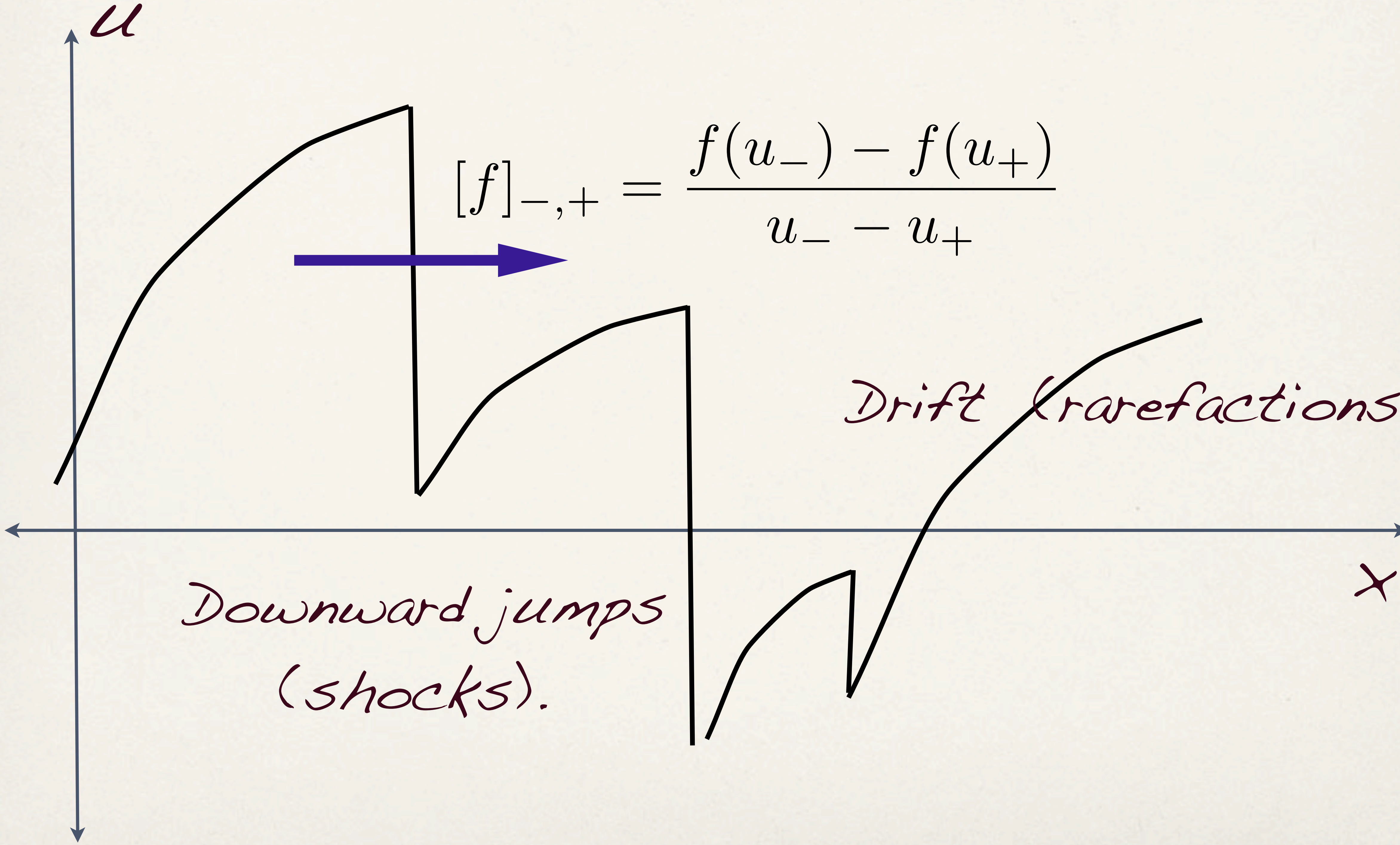
Markov processes and their generators

A Markov process is characterized by its transition semigroup Q and generator $\mathcal{A}$. For suitable test functions, we have

$$\mathcal{A}\varphi = \lim_{h \downarrow 0} \frac{Q_h \varphi - \varphi}{h}.$$

Typical profile of entropy solutions:

Bounded variation + downward jumps



$$[f]_{-,+} = \frac{f(u_-) - f(u_+)}{u_- - u_+}$$

Generators of spectrally negative Markov processes

A Feller process with BV sample paths and only downward jumps has an infinitesimal generator

$$\mathcal{A}\varphi(u) = \underbrace{b(u)}\,\varphi'(u) + \int_{-\infty}^{u} \underbrace{n(u,v)}\,(\varphi(v) - \varphi(u))\,dv.$$

Drift at level u. (rarefactions)

Jumps from u to v. (shocks)

Closure theorem (Srinivasan's thesis, 2009)

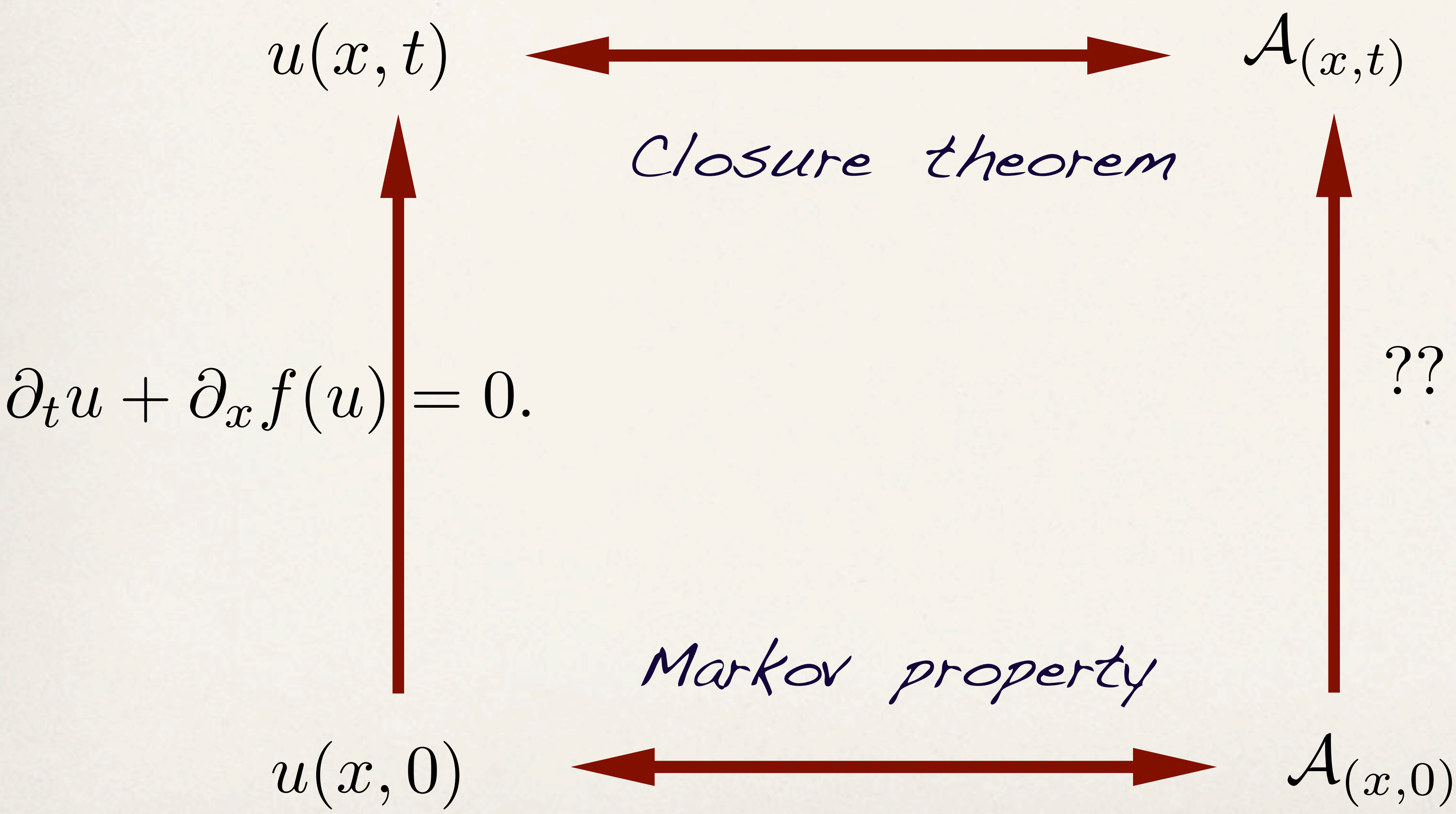
$$\partial_t u + \partial_x f(u) = 0.$$ f convex, differentiable.

Thm. 1. Assume the initial velocity $u(x,0)$ is a Markov process (in x) with only downward jumps.

Then so is the solution $u(x,t)$ for every $t>0$.

Proof follows earlier work of Bertoin, Groeneboom, but the generality of the result is important.

Since the process is Markov, it has an infinitesimal generator that depends on (x,t). Conceptually, we have the following picture.



The `generator` in time

First recall the definition of the generator:

$$\mathcal{A}\varphi(u) = b(u)\varphi'(u) + \int_{-\infty}^{u} n(u,v)\left(\varphi(v) - \varphi(u)\right)\, dv.$$

Now define an associated operator (here f is the flux function in the scalar conservation law):

$$\begin{aligned}\mathcal{B}\varphi(u) &= -f'(u)b(u)\varphi'(u) \\ &\quad - \int_{-\infty}^{u} \frac{f(v) - f(u)}{v - u} n(u,v)\left(\varphi(v) - \varphi(u)\right)\, dv.\end{aligned}$$

The operator $\mathcal{B}$ corresponds to the stochastic process $u(x,t)$ now viewed as a process in t.

$f'(u)$ is the speed at level u, so we have

$\partial_t u = -f'(u)b(u)$ for evolution by the drift.

Similarly, $\dfrac{f(u)-f(v)}{u-v}$ is the speed of a shock connecting levels u and v.

Can be obtained from Ito's formula (with jumps).

The backward Kolmogorov equations

Consider the backward equations associated to Markov process. Since we have a two-parameter process, we obtain <u>two</u> backward equations.

$$\partial_x \varphi + \mathcal{A}\varphi = 0. \quad \text{and} \quad \partial_t \varphi + \mathcal{B}\varphi = 0.$$

This is formal, but....

The Lax equation

Since we have semigroups in x and t, we impose compatibility. That is, we require

$$\partial_t \partial_x \varphi = \partial_x \partial_t \varphi$$

If this holds for a large enough class of functions we obtain the Lax equation

$$\partial_t \mathcal{A} - \partial_x \mathcal{B} = [\mathcal{A}, \mathcal{B}].$$

Main results for stationary statistics.

$u(x,t)$ ⟷ (Closure theorem) ⟷ $\mathcal{A}(t)$ ⟷ (Characterization of generator) ⟷ (b_t, n_t)

↑ $\partial_t u + \partial_x f(u) = 0.$ ↑ $\dot{\mathcal{A}} = [\mathcal{A}, \mathcal{B}]$ ↑ Kinetic equations.

$u(x,0)$ ⟷ $\mathcal{A}(0)$ ⟷ (b_0, n_0)

The Lax equation may also be derived via:

2. A Boltzmann-type equation using the evolution of a single shock and rarefaction wave.

3. Vol'pert's BV chain rule, the Markov property and an unjustified interchange of limits.

4. Hopf's method: a formal evolution equation for the Fourier transform of the law of $u(x,t)$.

Still no proof (in progress, Mattingly, Srinivasan).

Kinetic equations of shock clustering for arbitrary f

$$\partial_t b = -f''(u) b^2.$$

Rarefactions

Shocks

$$\partial_t n + \partial_u (n V_u) + \partial_v (n V_v) \\ = Q(n,n) + n\left(\left([f]_{u,v} - f'(u)\right)\partial_u b - b f''(u)\right).$$

$$Q(n,n)(u,v) = \int_v^u \left([f]_{u,w} - [f]_{w,v}\right) n(u,w) n(w,v)\, dw \\ - \int_{-\infty}^u \left([f]_{u,v} - [f]_{v,w}\right) n(u,v) n(v,w)\, dw \\ - \int_{-\infty}^u \left([f]_{u,w} - [f]_{u,v}\right) n(u,v) n(u,w)\, dw.$$

Collision kernel

Exact solutions to Burgers turbulence, part 1.

Main assumptions:

Burgers equation + Levy process initial data

L. Carraro and J. Duchon, C.R. Acad. Sci., 319, (1994)
J. Bertoin, Commun. Math. Phys. 193 (1998).

The kinetic equation for clustering (Burgers)

$$\dot{b} = -b^2, \qquad \partial_t n(u,v,t) = \underbrace{D(b,n)}_{\text{Drift}} + \underbrace{Q(n,n)}_{\text{Collisions}}.$$

$$D(b,n) = \left(\frac{u-v}{2}\right)\left(b(u)\partial_u n - \partial_v(b(v)n)\right)$$

$$Q(n,n) = \frac{u-v}{2}\int_v^u n(u,w)n(w,v)\,dv \quad \longleftarrow \text{Birth}$$

$$-n(u,v)\int_{-\infty}^v n(v,w)\left(\frac{u-w}{2}\right)dw$$

$$-n(u,v)\int_{-\infty}^u n(u,w)\left(\frac{w-v}{2}\right)dw$$

Death (last two terms)

Generators of Levy processes

For Levy processes the transition probabilities are independent of the state, and we have

$$n(u,v) = n(u-v).$$

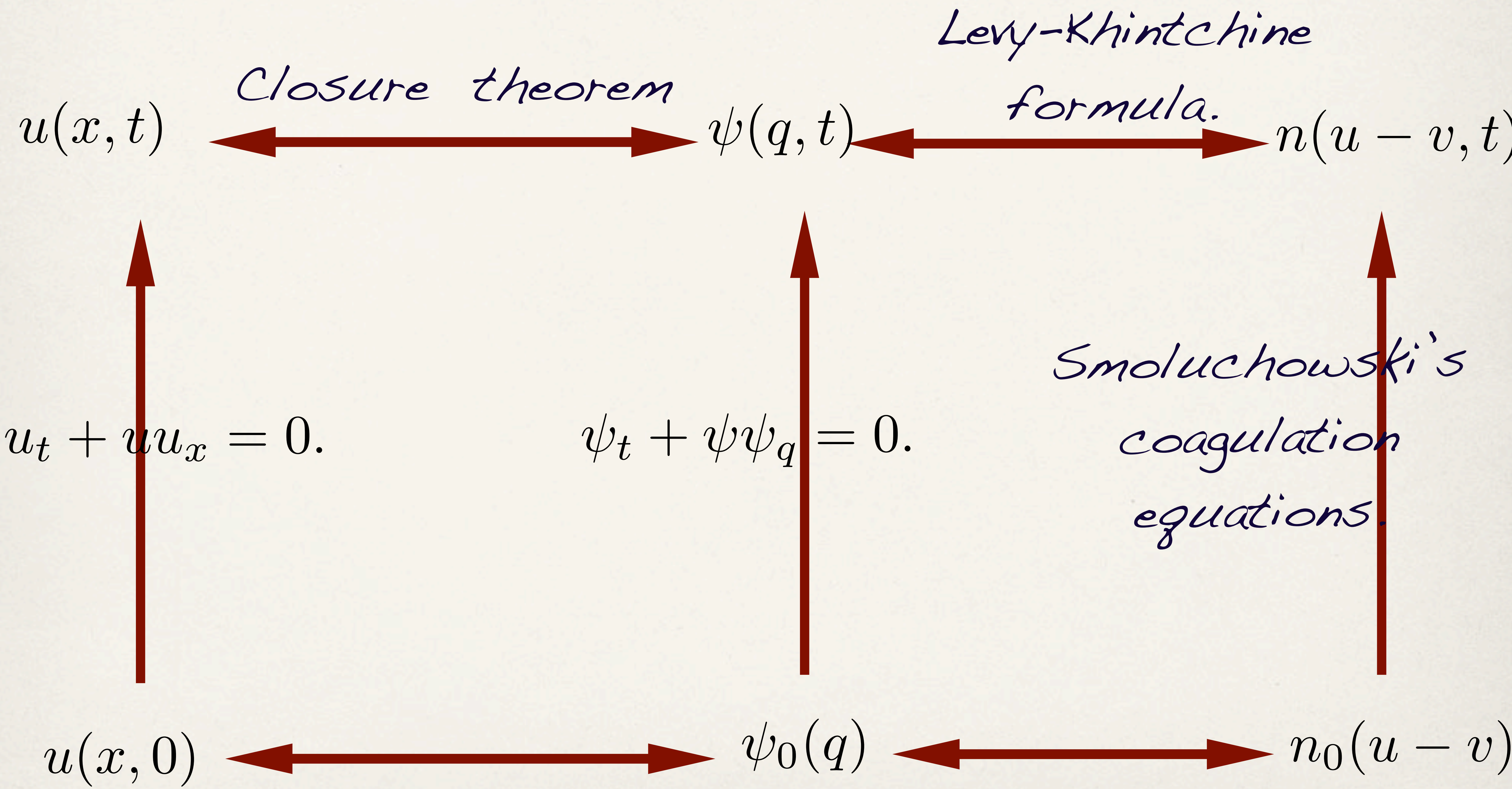
Now use Fourier analysis to find

$$\mathcal{A}e^{iks} = \psi(ik)e^{iks}, \quad k \in \mathbb{R}.$$

For example, if $u(x)$ is a Brownian motion,

$$\psi(ik) = -\frac{k^2}{2}.$$

Burgers equation and Levy processes.



Exact solutions to Burgers turbulence, part 2.

Main assumptions:

Burgers equation + white noise initial data

P. Groeneboom, PTRF, 81, (1989)

L. Frachebourg, P.A. Martin, J. Fluid Mech., 417, (2000).

The kinetic equation for clustering (Burgers)

$$\dot{b} = -b^2, \qquad \partial_t n(u,v,t) = \underbrace{D(b,n)}_{\text{Drift}} + \underbrace{Q(n,n)}_{\text{Collisions}}.$$

$$D(b,n) = \left(\frac{u-v}{2}\right)\left(b(u)\partial_u n - \partial_v (b(v)n)\right)$$

$$Q(n,n) = \frac{u-v}{2}\int_v^u n(u,w)n(w,v)\,dv \quad \longleftarrow \text{Birth}$$

$$-n(u,v)\int_{-\infty}^v n(v,w)\left(\frac{u-w}{2}\right)dw$$

$$-n(u,v)\int_{-\infty}^u n(u,w)\left(\frac{w-v}{2}\right)dw$$

Death (the last two terms)

Groeneboom's solution (Burgers with white noise)

$$b(u,t) = \frac{1}{t}, \quad n(u,v,t) = \frac{1}{t^{1/3}} n_* \left(\frac{u}{t^{2/3}}, \frac{v}{t^{2/3}} \right).$$

The jump density factorizes as:

$$n_*(u,v) = \frac{J(v)}{J(u)} K(u-v),$$

where J and K have Laplace transforms:

$$j(q) = \frac{1}{Ai(q)}, \quad k(q) = -2 \frac{d^2}{dq^2} \log Ai(q).$$

This was not how Groeneboom found his solution!

In order to verify that it solves the kinetic equations we need to use some interesting identities. These are best written in terms of the variable $e = j'/j$. Then

$$e' = -q + e^2,$$ ← Riccati eqn.

$$k' = -2(1 - ek),$$

$$k''' = 6kk' + 4qk' + 2k.$$

These yield three moment identities, such as

$K * J(x) = x^2 J - J'$ and some amazing cancellations.

The Painleve property

In fact, e is the first Airy solution to Painleve 2.

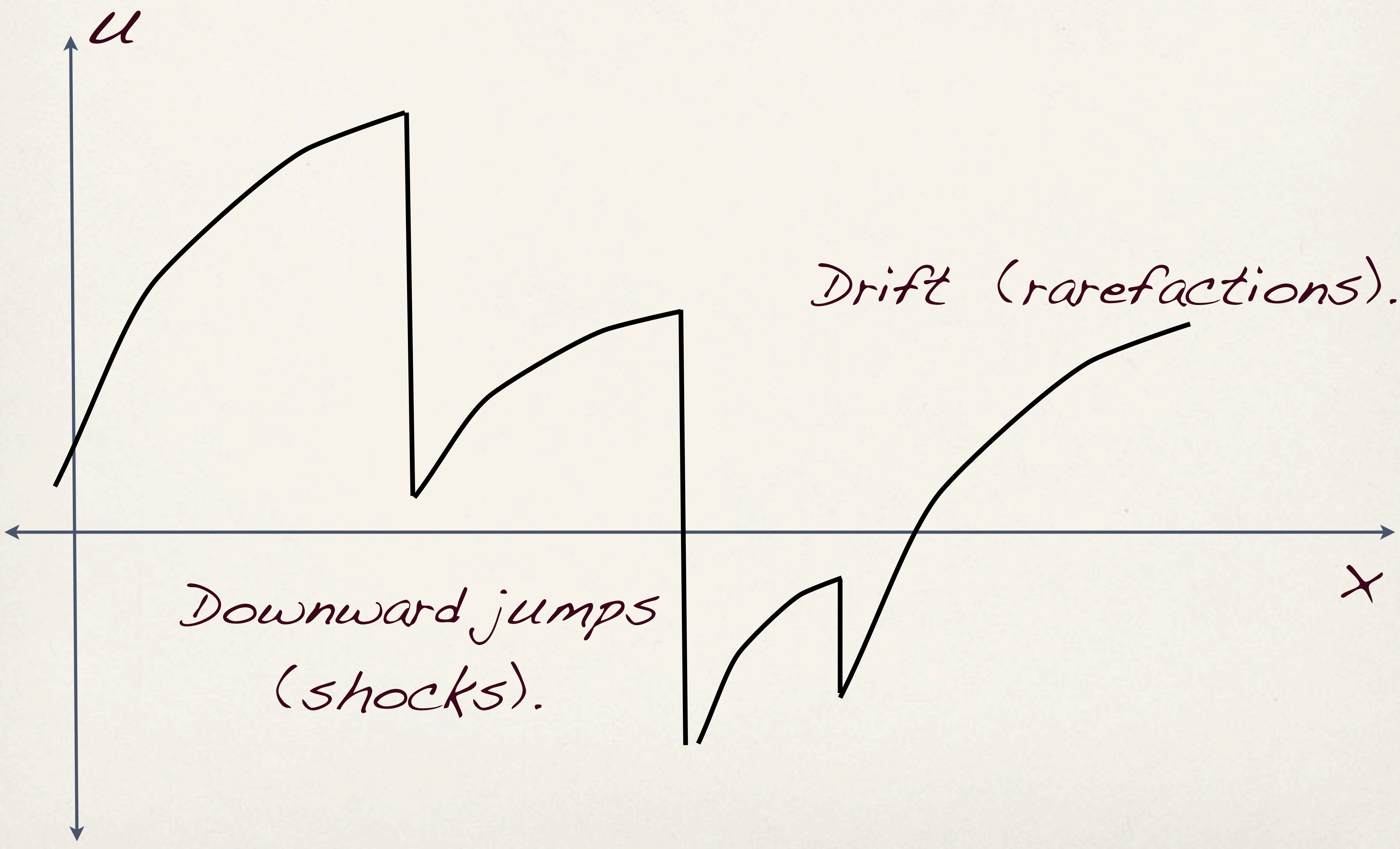
$$w'' = 2w^3 + 2wq + \frac{1}{2}.$$

This solution to Painleve-2 is 'simpler' than the solution for the Tracy-Widom GUE law.

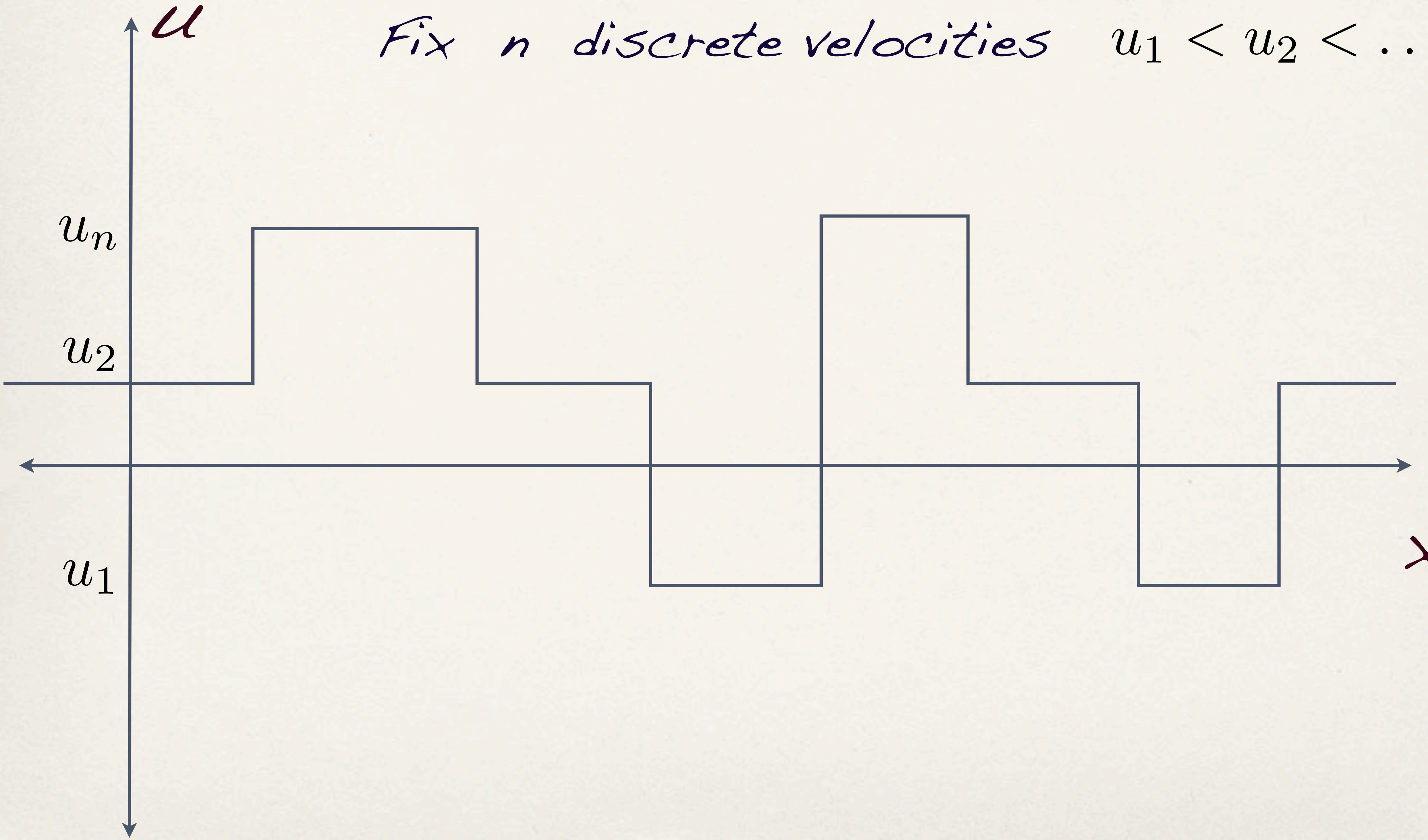
It also arises in a different manner. Here it corresponds to the generator of a Markov process.

Hamiltonian structure, flows of Markov operators, and complete integrability.

Markov processes with downward jumps



Continuous time Markov chain with discrete states.



Generators of the discretized process

The generator A is an $n \times n$ matrix with positive entries on the off-diagonal. The diagonal entries are determined by setting the sum of each row to zero.

If we don't require positivity, the set of such matrices is a Lie algebra. Let's call the associated group the <u>Markov group</u>. Of course, positivity is needed for a probabilistic interpretation.

The discrete Lax equations

$$\dot{A} = [A, B]$$

$$B_{ij} = F_{ij} A_{ij}, \quad i \neq j.$$

$$B_{ii} = -\sum_{j \neq i} B_{ij}.$$

$$F_{ij} = \frac{f(u_i) - f(u_j)}{u_i - u_j}, \quad i \neq j.$$

Complete integrability

Thm. 2. The discrete Lax equations are a completely integrable Hamiltonian system.

The linearization can be computed by a matrix factorization, and is expressed in terms of theta functions (in principle, and not yet in practice!).

Particle on a sphere (Neumann)
Geodesic flow on ellipsoids (Jacobi)

Moser, Adler-van Moerbeke

Rigid body dynamics (Euler).
Geodesic flows on so(n)
(Manakov).

$$\dot{A} = [A, B]$$

Random matrix theory

o.d.e for Fredholm determinants
Jimbo-Miwa-Mori-Sato,
Harnad-Tracy-Widom.

Burgers turbulence,
Kinetic theory of shocks.

Flows of Markov processes

Proof of Theorem 2.

Hamiltonian structure: Split gl(n) into diagonal + Markov, and use Kirillov's equation.

Complete integrability: Split loop group over gl(n) using similar decomposition of gl(n). Follows immediately from Adler-Kostant-Symes theorem.

With hindsight, the discrete system is a textbook example of the AKS theorem. Its remarkably similar to Manakov's integration of Euler equations, except that the flow is on a 'probabilistic group' instead of $so(n)$.

The catch: We don't understand the large n limit, and the discrete system doesn't have a clear probabilistic interpretation yet.

Summary

1. Entropy solutions to scalar conservation laws preserve spectrally negative Markov processes.

2. The evolution of generators is given by a Lax pair obtained from compatibility of backward equations.

3. This yields a kinetic theory of shock clustering.

4. The Lax equation is a completely integrable Hamiltonian system with a quadratic Hamiltonian, on the (formal) Lie algebra of generators.

MS 57, 65 : Stochastic dynamics and coherent structures

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