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Numerical Relativity:
On overview of the field and recent results on black hole simulations

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Gravitational wave detectors

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A global network of gravitational wave detectors in now either in an advanced state of construction, or actually taking data!

The collision of compact objects (black holes, neutron stars) is considered one of the most promising sources for detection in the next few years.


TAMA,
Tokio


LIGO,Washington

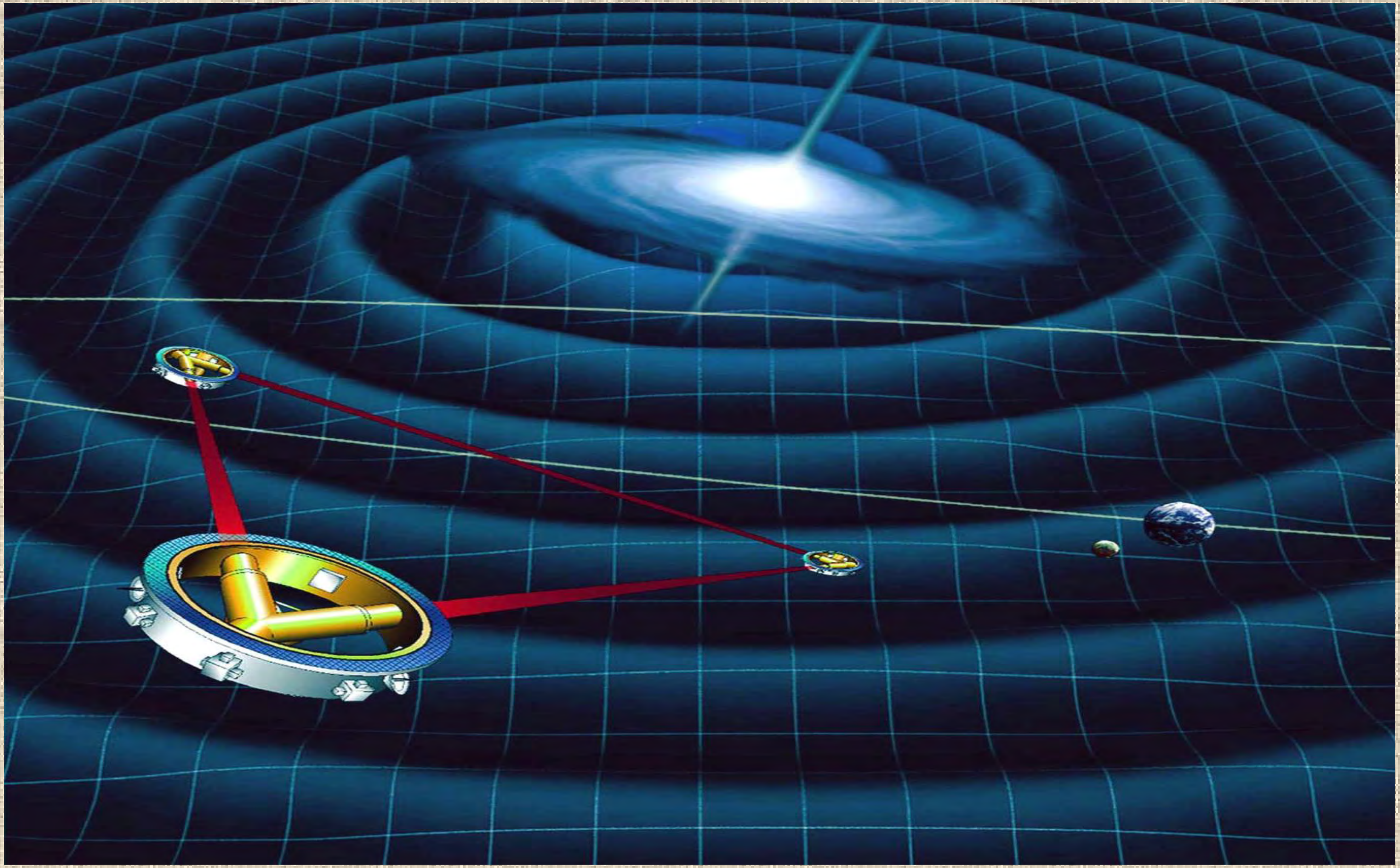

VIRGO, Pisa


LIGO, Louisiana


GEO 600, Hanover

The future: LISA

(Laser Interferometer Space Antenna)



Einstein’s field equations

The dynamics of the gravitational field are described by the Einstein field equations:

$$\boxed{G \qquad 8}$$

$$0,1,2,3$$

These equations relate the geometry of space-time (the left hand side) with the distribution of mass and energy (the right hand side).

Einstein’s equations form a system of 10 non-linear, coupled, partial differential equations in 4 dimensions.

Written on a general coordinate system they can have thousands of terms!

Numerical relativity

There are books full of exact solutions to Einstein's equations, but few of those solutions have a clear astrophysical interpretation. Exact solutions are typically found by asking for space-time to have a high degree of symmetry:

- Schwarzschild black hole: Static and spherically symmetric.
- Kerr black hole: Stationary and axially symmetric.
- Cosmology: isotropic and/or homogeneous.

When one wishes to study more complex systems, with astrophysical relevance (gravitational collapse, supernovae, collisions of compact objects) it is extremely difficult, or even impossible, to find exact solutions to Einstein's equations.

Numerical relativity tries to solve Einstein's equations using numerical approximations and computers.

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3+1 decomposition

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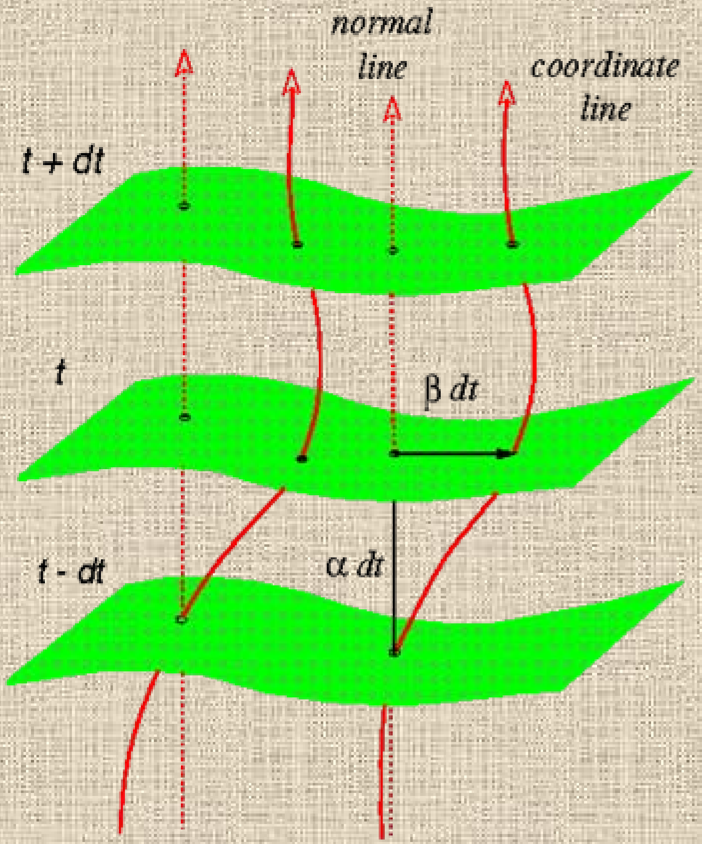
Most (but by no means all) of numerical relativity uses the 3+1 decomposition:



The 4-metric is rewritten as:

$$ds^2 = \left(-\alpha^2 + \beta_i\beta^i\right) dt^2 + 2\beta_i\, dt\, dx^i + \gamma_{ij}\, dx^i dx^j$$

- (lapse function): measures proper time between adjacent hypersurfaces.

$$d\tau = \alpha(t, x^i)\, dt$$

- ^{i} (shift vector): relates spatial coordinates between adjacent hypersurfaces.

$$x^i_{t+dt} = x^i_t - \beta^i(t, x^j)\, dt$$

- $_{ij}$ (spatial metric): measures distances within spatial hypersurfaces.

$$dl^2 = \gamma_{ij}\, dx^i dx^j$$

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Alternative formalisms

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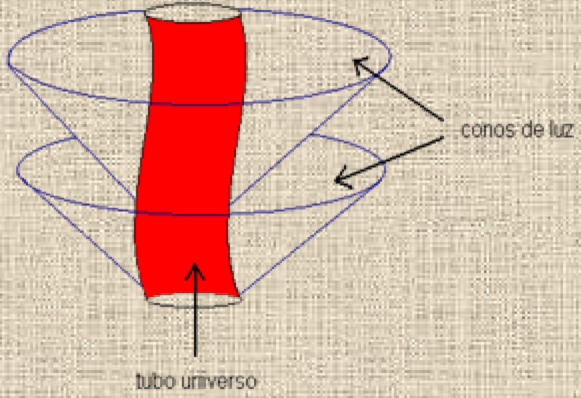
The 3+1 formalism is not the only way one can foliate spacetime for numerical purposes. Two alternatives are:

Characteristic formalism

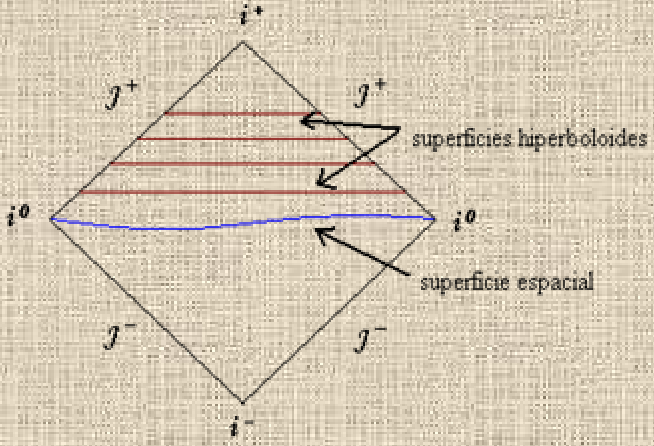
Spacetime is foliated by null surfaces starting from a central world tube. Compactification allows to reach null infinity in a finite computational domain.



J. Winicour, Liv. Rev. 2001 (gr-qc/0102085)

Conformal formalism

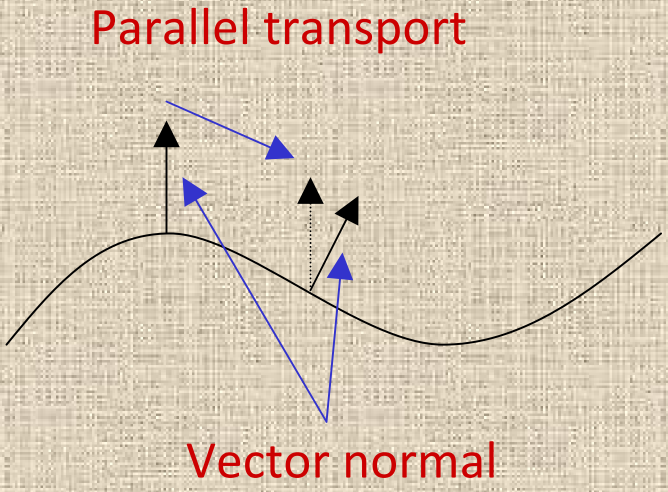
Hiperbolidal hypersurfaces are locally spacelike but reach null infinity. Also allows compactification.



H. Friedrich, Lec. N. Phys. 604 (2002) 1 (gr-qc/0209018),
J. Frauendiener, Lec. N. Phys. 604 (2002) 261 (gr-qc/0208093)
S. Husa, Lec. N.Phys. 604 (2002) 239-260 (gr-qc/0204043)

Extrinsic curvature

In the 3+1 formalism, the spatial metric describes the intrinsic geometry of the spatial hypersurfaces. In order to describe how those hypersurfaces are immersed in space-time one needs to introduce the "extrinsic" curvature:



The extrinsic curvature K_{ij} measures the change of the normal vector under parallel transport:

$$K_{\alpha\beta} := -\nabla_\alpha n_\beta$$

K_{ij} can be rewritten as:

$$K_{ij} = \frac{1}{2\alpha}\left[\partial_t \gamma_{ij} + \nabla_i^{(3)}\beta_j + \nabla_j^{(3)}\beta_i\right]$$

The last equation can be used to find the time derivative of the spatial metric (that is, it gives us an evolution equation for the spatial metric).

3+1 decomposition of Einstein’s equations

In the 3+1 formalism, Einstein’s equations are split in two groups by using the normal and parallel projections onto the spatial hypersurfaces:

- 4 equations with no time derivatives: “constraints”
- 6 equations with time derivatives: “evolution”

The constraints take the form:

- Hamiltonian:

$$R^{(3)} + (\mathrm{tr}\,\mathrm{K})^2 - K_{ij}K^{ij} = 16\pi\rho$$

- Momentum:

$$\nabla_j \left[K^{ij} - \gamma^{ij}\mathrm{tr}\,\mathrm{K}\right] = 8\pi j^i$$

The evolution equations

The evolution equations in 3+1 form are:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + \nabla_i^{(3)} \beta_j + \nabla_j^{(3)} \beta_i$$

$$\begin{aligned} \partial_t K_{ij} \quad = \quad & \beta^a \nabla_a K_{ij} + K_{ia} \nabla_j \beta^a + K_{ja} \nabla_i \beta^a \\ & - \nabla_i \nabla_j \alpha + \alpha \left[R_{ij}^{(3)} - 2 K_{ia} K_j^a + K_{ij} \, \mathrm{tr} \, \mathrm{K} \right] \\ & + 4\pi\alpha \left[\gamma_{ij} \left(\mathrm{tr} \, \mathrm{S} - \rho \right) - 2 S_{ij} \right] , \end{aligned}$$

These are called the “Arnowitt-Deser-Misner” (ADM) equations.

The ADM equations are not unique!

One can always add arbitrary multiples of the constraints (i.e. of zero) to obtain new evolution equations that are just as valid, The new equations will have the same physical solutions, but different mathematical properties (in particular, different stability properties).

The ADM evolution equations are NOT unique. One can add arbitrary multiples of constraints to them to obtain new evolution equations that are just as valid physically.

The new equations will have the same physical solutions, but different mathematical properties and different stability properties.

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Hyperbolicity and well posedness

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Consider a system of equations for some dynamical variables u^i of the form:

$${}_t u^i \qquad M^x_{ij} \; {}_x u^j \qquad M^y_{ij} \; {}_y u^j \qquad M^z_{ij} \; {}_z u^j$$

Construct the “principal symbol” $S \quad M^a v_a$ with v_a an arbitrary (unit) vector.

The system is called “hyperbolic” if all eigenvalues of *S* are real for arbitrary v_a. Furthermore, the system is called “strongly hyperbolic” if *S* has a complete set of eigenvectors for all v_a, and “symmetric hyperbolic” if *S* can be diagonalized in a way that is independent of v_a.

Symmetric hyperbolic systems can be shown to be “well posed”, that is, unique solutions exist (at least locally) and those solutions change continuously with the initial data. Strongly hyperbolic systems are also well posed modulo some extra smoothness conditions and boundary issues.

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Reformulations of the evolution equations

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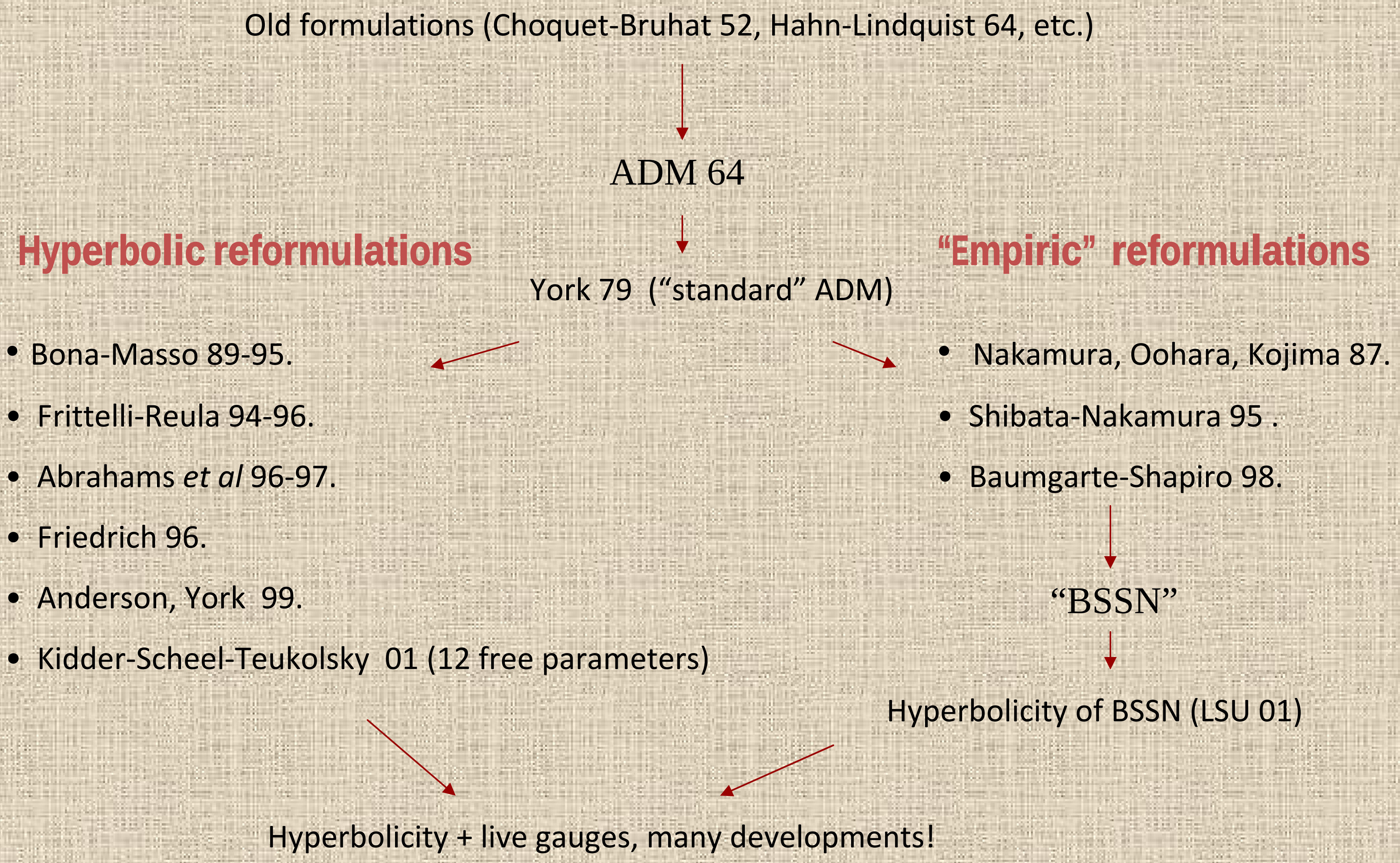


Old formulations (Choquet-Bruhat 52, Hahn-Lindquist 64, etc.)

ADM 64

York 79 (“standard” ADM)

Hyperbolic reformulations

- Bona-Masso 89-95.
- Frittelli-Reula 94-96.
- Abrahams *et al* 96-97.
- Friedrich 96.
- Anderson, York 99.
- Kidder-Scheel-Teukolsky 01 (12 free parameters)

“Empiric” reformulations

- Nakamura, Oohara, Kojima 87.
- Shibata-Nakamura 95 .
- Baumgarte-Shapiro 98.

“BSSN”

Hyperbolicity of BSSN (LSU 01)

Hyperbolicity + live gauges, many developments!

Choosing the gauge

The Einstein equations provide us with evolution equations for the spatial metric $_{ij}$ and the extrinsic curvature K_{ij}.

What about the evolution of the lapse and the shift ^{i}?

The Einstein equations say nothing about them. These quantities can in fact be specified freely, since they describe the evolution of the coordinate system, which is completely arbitrary.

The lapse and shift are known as the “gauge” and specifying them is known as “choosing (or fixing) the gauge”. However, we can't just specify them as known functions of space-time for a very simple reason: Which functions are a good choice?

The lapse and shift must be chosen dynamically as functions of the geometry, *i.e.* we construct the coordinates as we go along.

Slicing conditions

How to choose the lapse function? Some possibilities are:

- geodesic slicing: = 1

Normal lines to the hypersurfaces are geodesics (free fall observers). This is a very bad choice since geodesics can easily focus producing coordinate singularities (caustics).

- maximal slicing: tr*K* = 0

Volume elements remain constant, so no focusing can occur. This is a very common choice and implies that the lapse function must obey the following elliptic equation:

Disadvantage: elliptic equations are hard to solve!

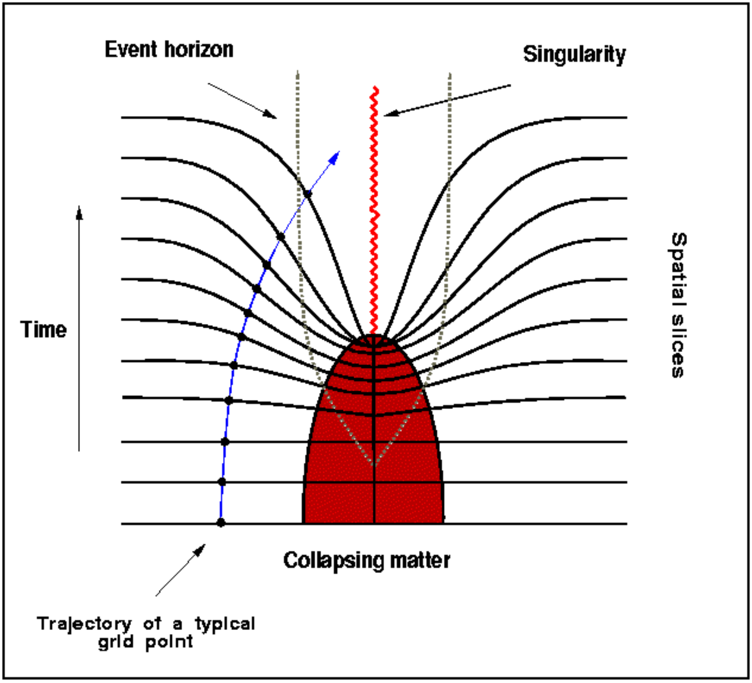
$$\nabla^2 \alpha = \alpha \left[K_{ij} K^{ij} + 4\pi \left(\rho + S \right) \right]$$

- algebraic slicings:

The lapse, or its time derivative, are given directly as functions of the geometric variables.

Singularity avoiding slicings

Some slicing conditions have the property of slowing down time in a region that approaches a singularity (maximal slicing has this property).



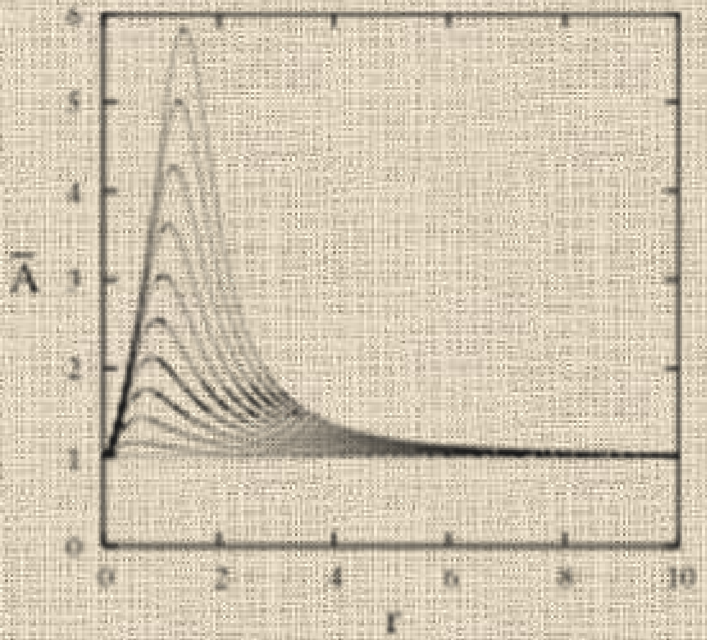
SINGULARITY AVOIDING SLICING

One disadvantage of these type of slicings is the following:

They produce a large stretching of the spatial hypersurfaces in the region close to the singularity.

This effect shows up as a very rapid growth of the radial metric componets that eventually causes a numerical code to fail due to lack of resolution.

Shift conditions



Evolving black holes with a vanishing shift causes the horizon to grow rapidly in coordinate space (since Eulerian observers keep falling in), which means that eventually all the computational domain ends up inside the black hole.

Worse still, the differential speed of infalling coordinate lines causes the radial metric to grow rapidly without bound: “slice stretching”.

For systems with angular momentum (rotating neutron stars or black holes), the dragging of inertial frames can be so severe that a non-zero rotational shift vector is *absolutely essential* in order to avoid having large shears developing in the spatial metric that will rapidly cause the simulation to crash. This type of situation also occurs in the case of orbiting compact objects.

Boundary conditions

Most physical systems under study extend all the way to infinity, but 3+1 simulations have a boundary at a finite distance (however far). One must therefore impose artificial boundary conditions there.

For a long time boundary conditions were not given the attention they deserved, as people considered themselves lucky if they could keep the boundaries stable.

This attitude has changed in the last few years, and today a lot of work is done in trying to ensure that the boundaries are compatible with Einstein’s equations (“constraint” preserving boundary conditions).

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Numerology of numerical relativity

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RAM: 500 x 500 x 500 grid points ~ 10^8 points

100 variables ~ 1 KB per point

Total: ~ 100 GB’s of RAM

FLOP’s: ~ 10,000 time steps

~ 10,000 FLOP´s per point per step

Total: ~ 10^4 x 10^4 x 10^8 = 10^{16} FLOP’s

Disk (storage): Output 10 variables every 100 steps (for analysis)

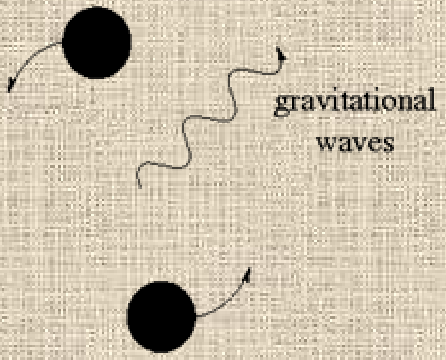
Total: ~ 1 Tbyte

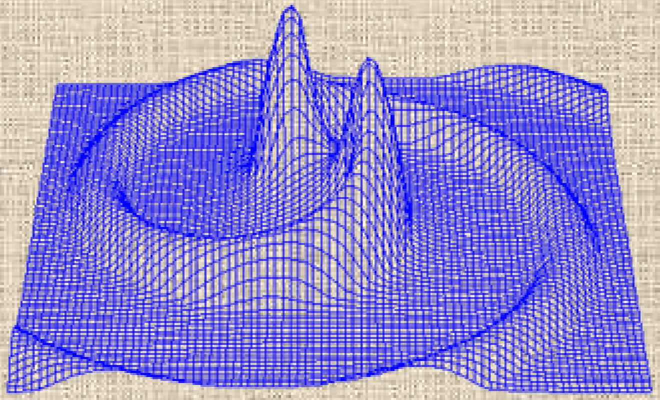
Zero law: Supercomputers are never big enough!

Black hole collisions:
the two body problem

The two body problem was solved in Newtonian gravity over 300 hundred years ago.

In general relativity, the simplest form of the two body problem is that of two orbiting black holes. This problem can not be solved analytically.





What is the problem?

Orbiting bodies emit gravitational waves: The system loses energy continuously and the orbit shrinks until the objects collide.

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Gravitational waves from orbiting BH binaries

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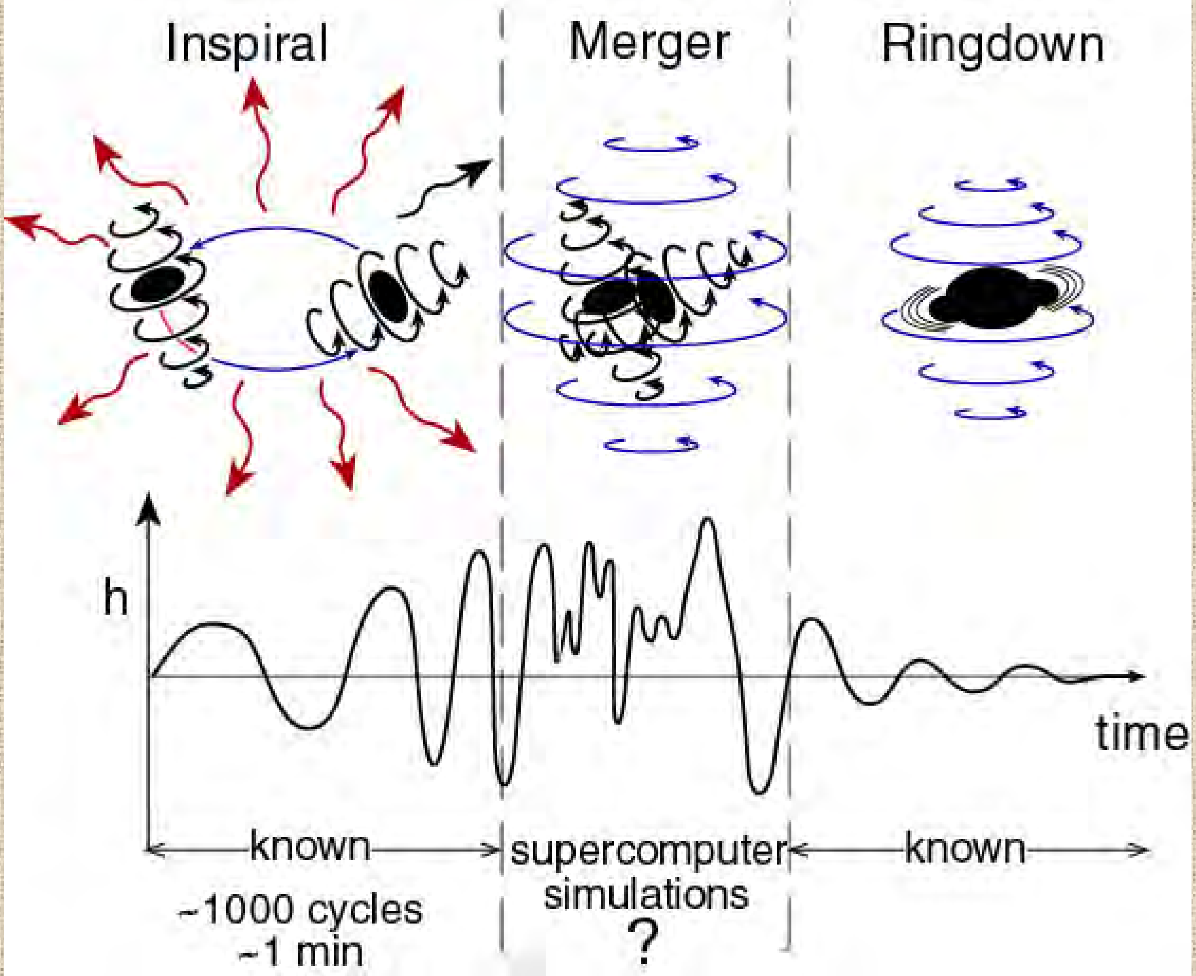

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Head-on collision of two black holes

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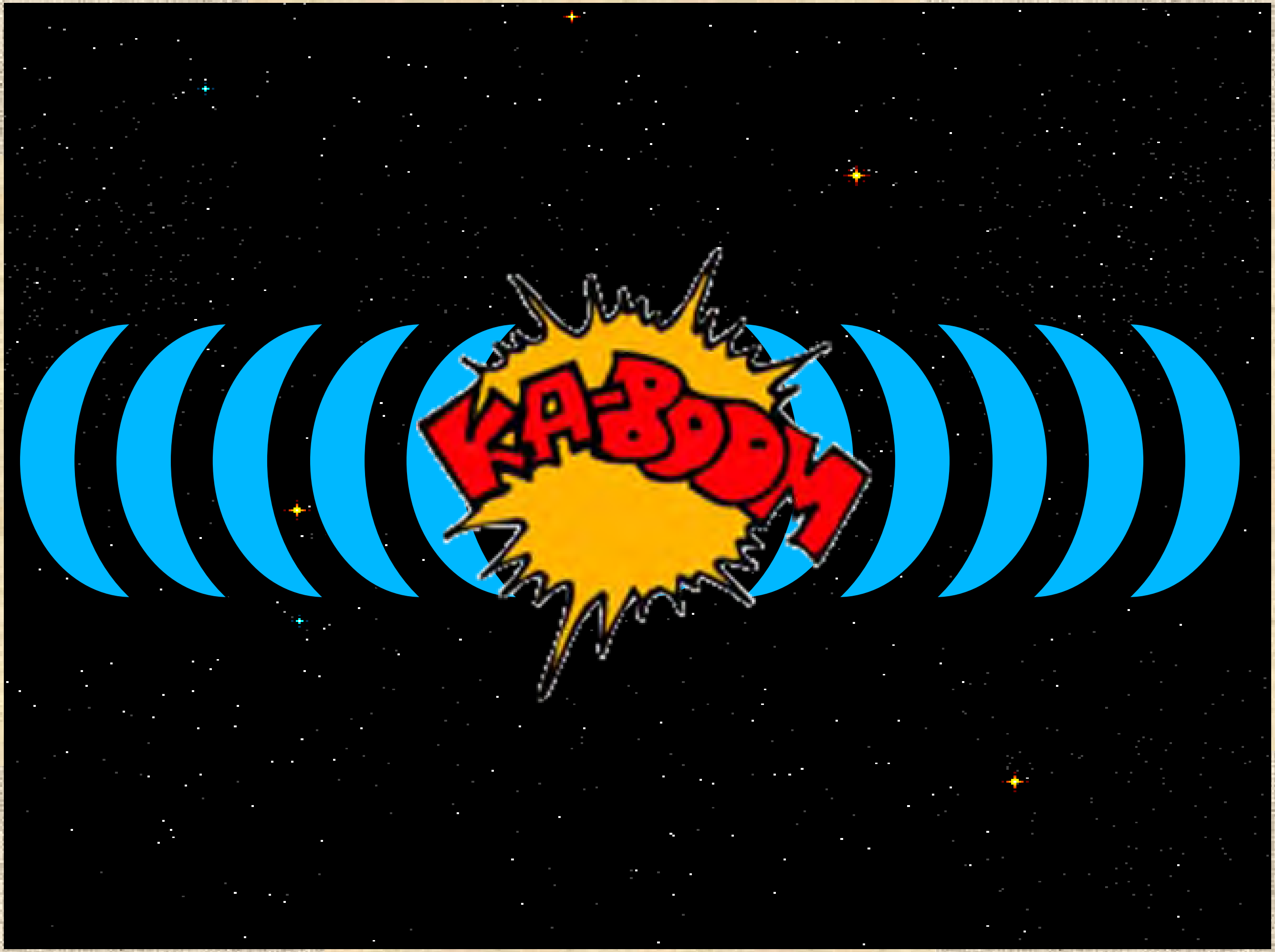

Ingredients that go into a succesful black hole evolution

- Good initial data.

- Well posed system of evolution equations: strongly or even symmetric hyperbolic (gauge choice is crucial here).

- A good gauge that keeps the situation roughly stationary.

- A way to handle the singularities (excision, punctures, ...)

- Of course, good boundary conditions are important ... .

- And of course, a robust code with stable numerical methods and high accuracy ....

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BH collision timeline: a 40 year effort!

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1962
ADM

1964
Hahn+Lindquist

1975
head-on collision
Smarr + Eppley

1978
Maximal slicing,
minimal distortion
Smarr+York

1979
York's paper

1984
Excision
suggested
by Unruh

1989-1995
hyperbolicity,
hyperbolic slicings
Bona+Masso
and others

1994
Cook initial data

1994-1998
GRAND CHALLENGE

1994-1995
Excision, 1+log,
head-on collision
(NCSA-WashU)

1997
Puncture data
Brandt-Bruegmann

1999
(since 1989)
BSSN

2000
Grazing
collisions

2001
Lazarus
waveforms!

2002
Hyperbolic
shifts

2002-2005
ISCO's
Pre-ISCO's
(corotation)

2003
First orbit

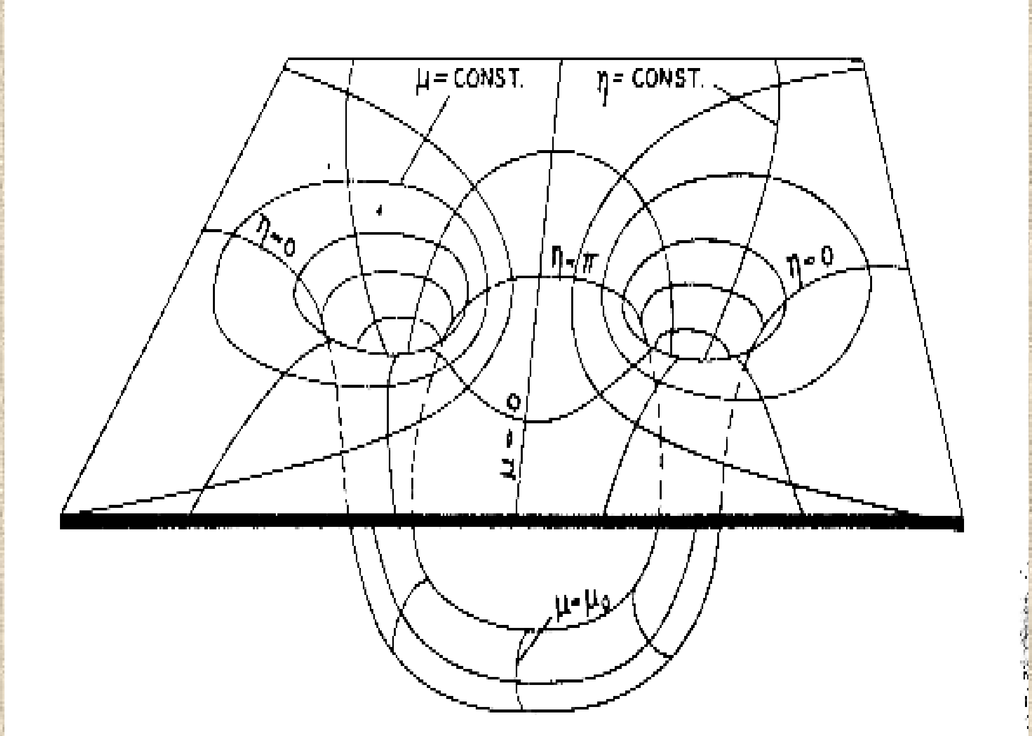
2005
First orbits!
F. Pretorius

2005-2006

Orbits with
puncture data

Hahn and Lindquist (1964)

- Head-on collision of two equal mass black-holes (two body problem in geometro-dynamics, the term “black hole” did not exist).
- Not ADM, 4-Christoffel symbols as main evolution variables.
- Time-symmetric initial data (single sheet).
- Coordinate singularities in metric factored out analytically.
- normal Gaussian coordinates (geodesic slicing, zero shift).



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Initial data for black holes

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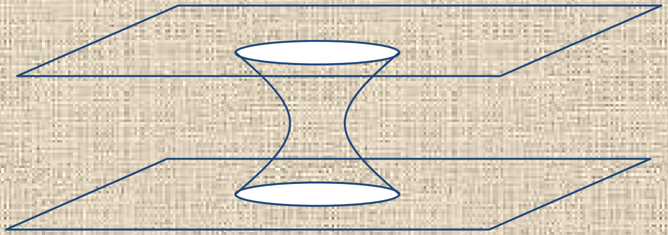
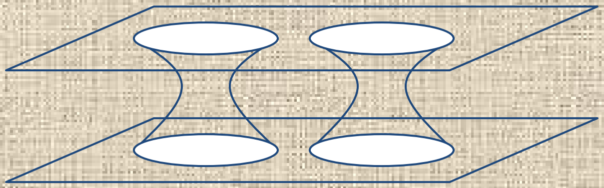
Initial data for black holes can be constructed using a "topological model" that consists on joining two separate universes through a "wormhole" (an Einstein-Rosen bridge).



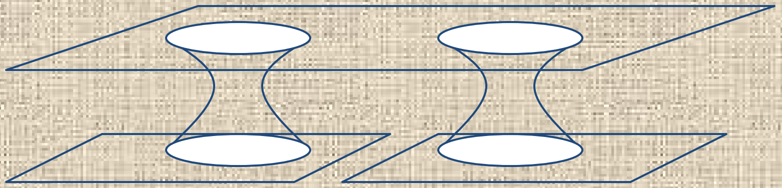
Schwarzschild

This idea comes from the fact that the Schwarzschild black hole solution has in fact such a wormhole inside it.



Misner

Data for multiple black holes can be obtained by adding more wormholes to join two or more "parallel" universes.



Brill-Lindquist

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Head-on collision of two black holes (NCSA 1995)

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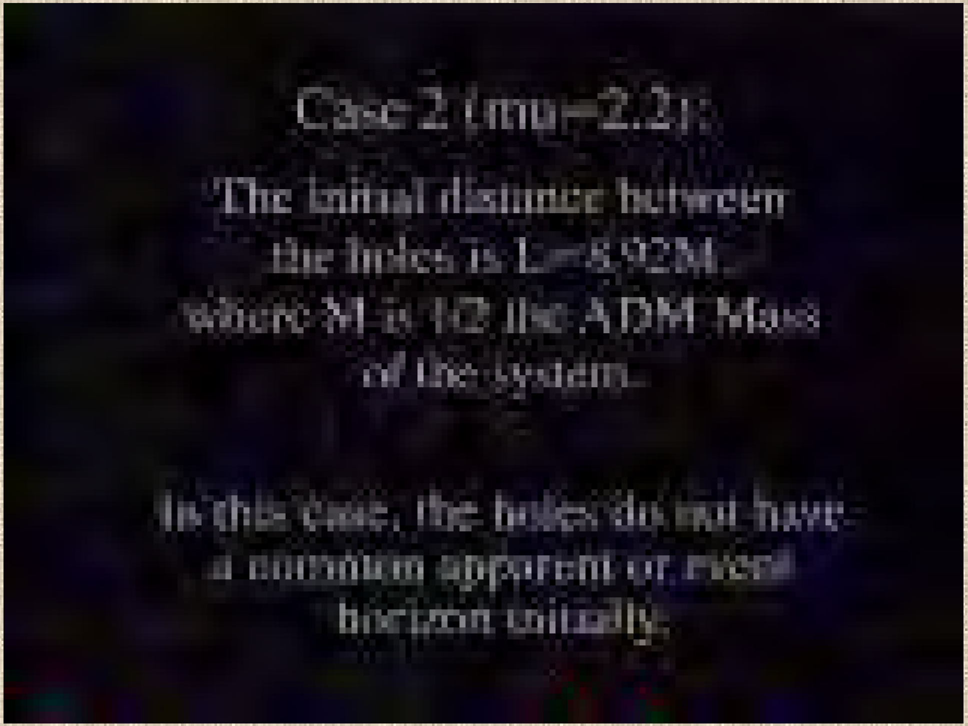
Two equal mass black holes initially at rest that fall into each other.

- The level surface shows the lapse function which is an indicator of the strength of the gravitational field.
- The color map shows the gravitational waves.



Case 2 (mu=2.2):

The initial distance between
the holes is L=8.92M,
where M is 1/2 the ADM Mass
of the system.

In this case, the holes do not have
a common apparent or event
horizon initially.

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Event horizons (NCSA 1995)

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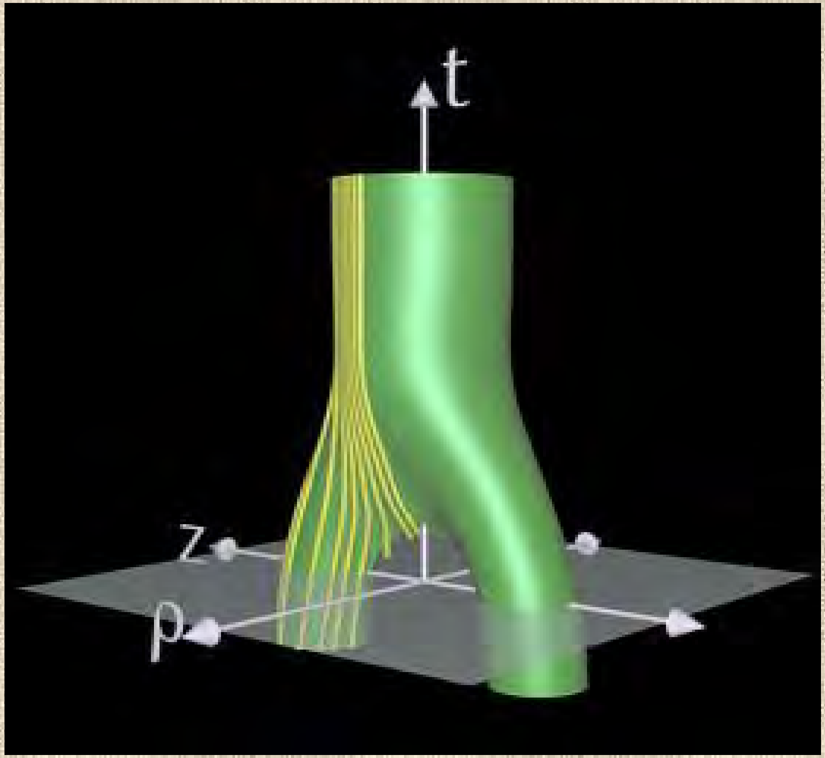
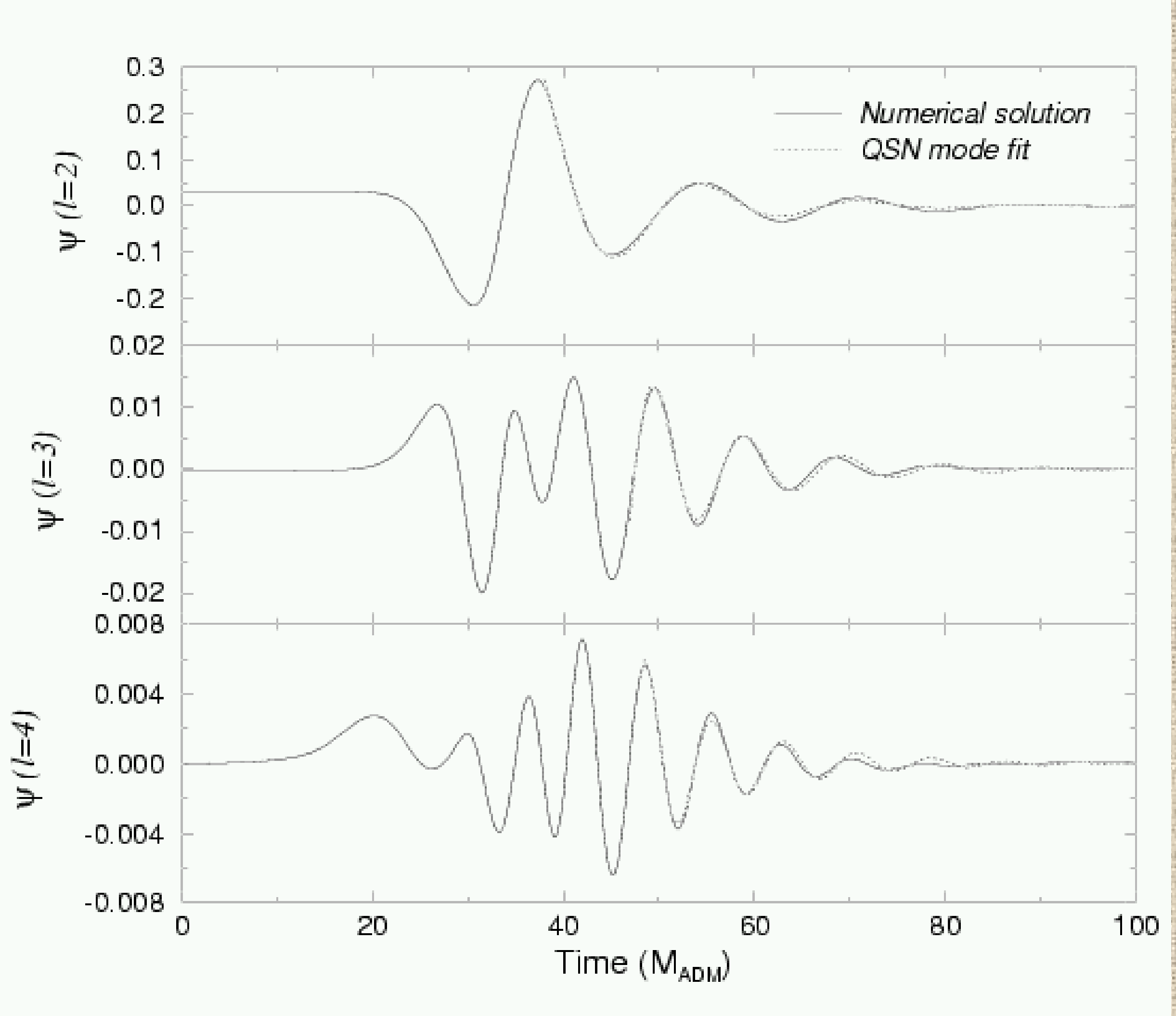
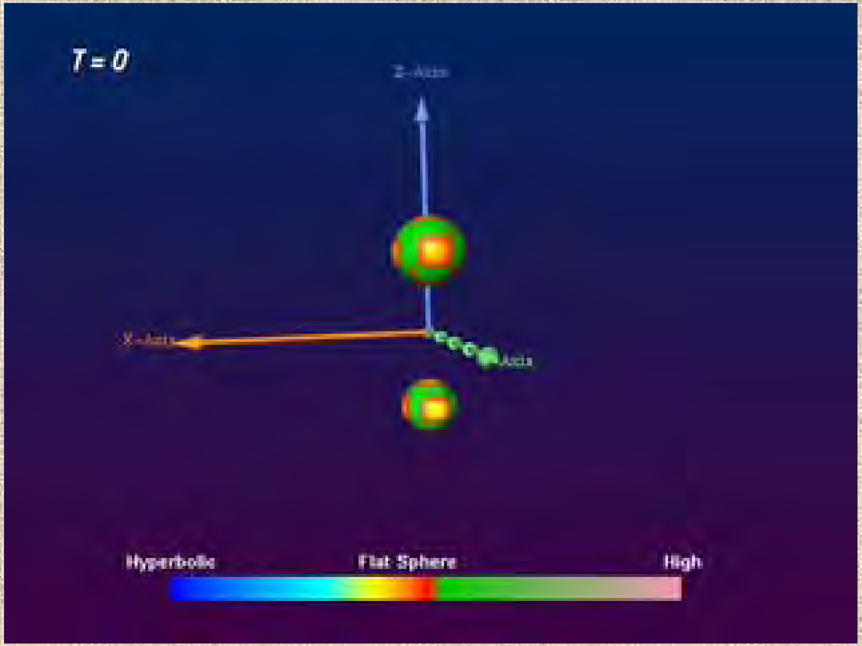


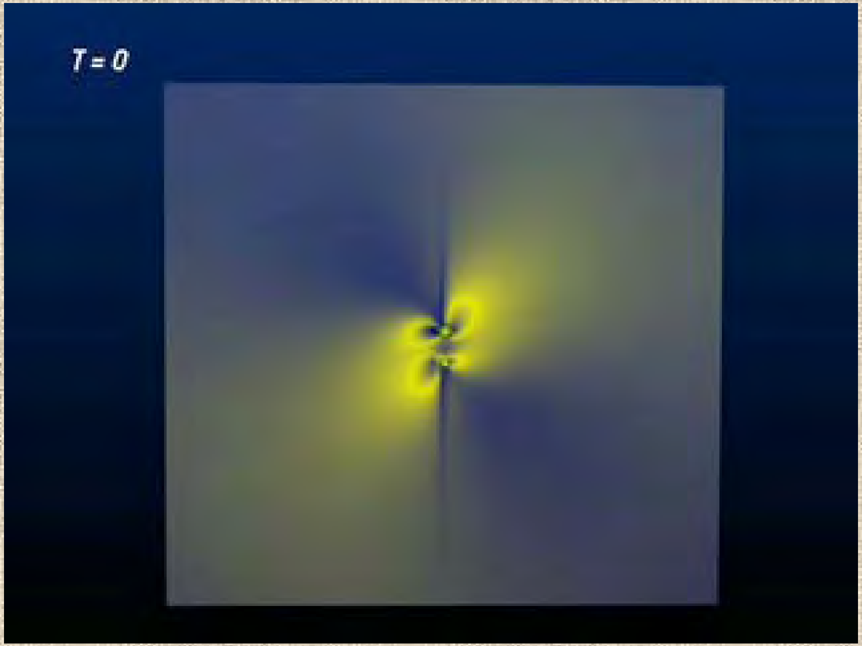
Evolution of the “event horizon” for a black hole collision. The event horizon marks the region of no return. The diagram shows the well known “pair of pants”.

Head-on collision: gravitational wave signal



Grazing collision (AEI-Potsdam 1999)





Left hand side: Evolution of apparent horizons. The color indicates the curvature (red is a sphere).

Right hand side: Gravitational waves (Newman-Penrose quantity $_4$). Large numerical error is evident toward the end.

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Pre-ISCO:
The Discovery Channel movie
(Potsdam 2002)

First Breakthrough: Frans Pretorius (2004)

In 2004 Pretorius used an approach that was very different to the standard:

- Not 3+1, evolve directly the 4-metric.
- Harmonic-type coordinates in time and space.
- Initial data: scalar field configurations that rapidly collapse to BH’s.
- Compactified spatial infinity.
- “Constraint damping”.
- Adaptive mesh.

All is very different, but it worked! It worked so well that several groups (including Pretorius) are using this technique today.

Second Breakthrough: Moving Punctures (2005)

The Goddard and Brownsville groups achieve multiple orbits with a 3+1 approach and rather standard techniques but a *key* modification. Other groups followed rapidly (Jena, PSU, AEI, LSU).

- BSSN formulation.
- 1+log slicing (Bona-Masso family).
- Gamma-driver shift condition with modifications.
- No excision and no puncture evolution: Allow the punctures to move by absorbing singularity into conformal factor. This seems to be the key idea which makes everything much simpler.
- High-resolution (4th order + AMR).

Multiple orbits + waveform extraction!

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Brownsville simulations

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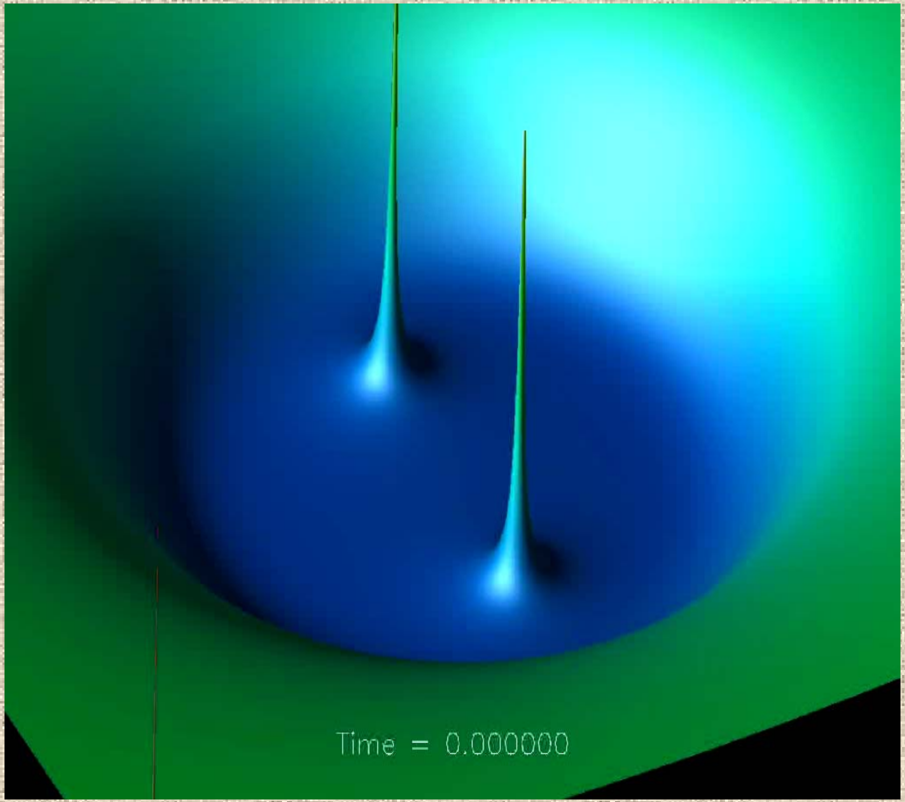
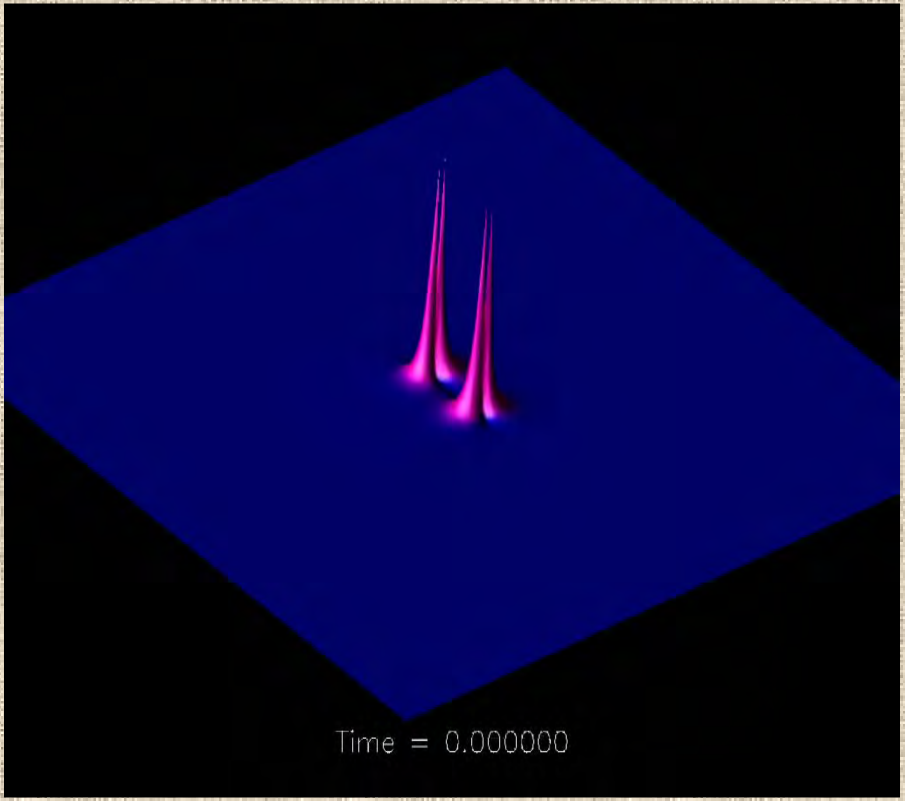


Evolution of conformal factor



Gravitational waves ($_4$)

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Univ. of Jena simulations

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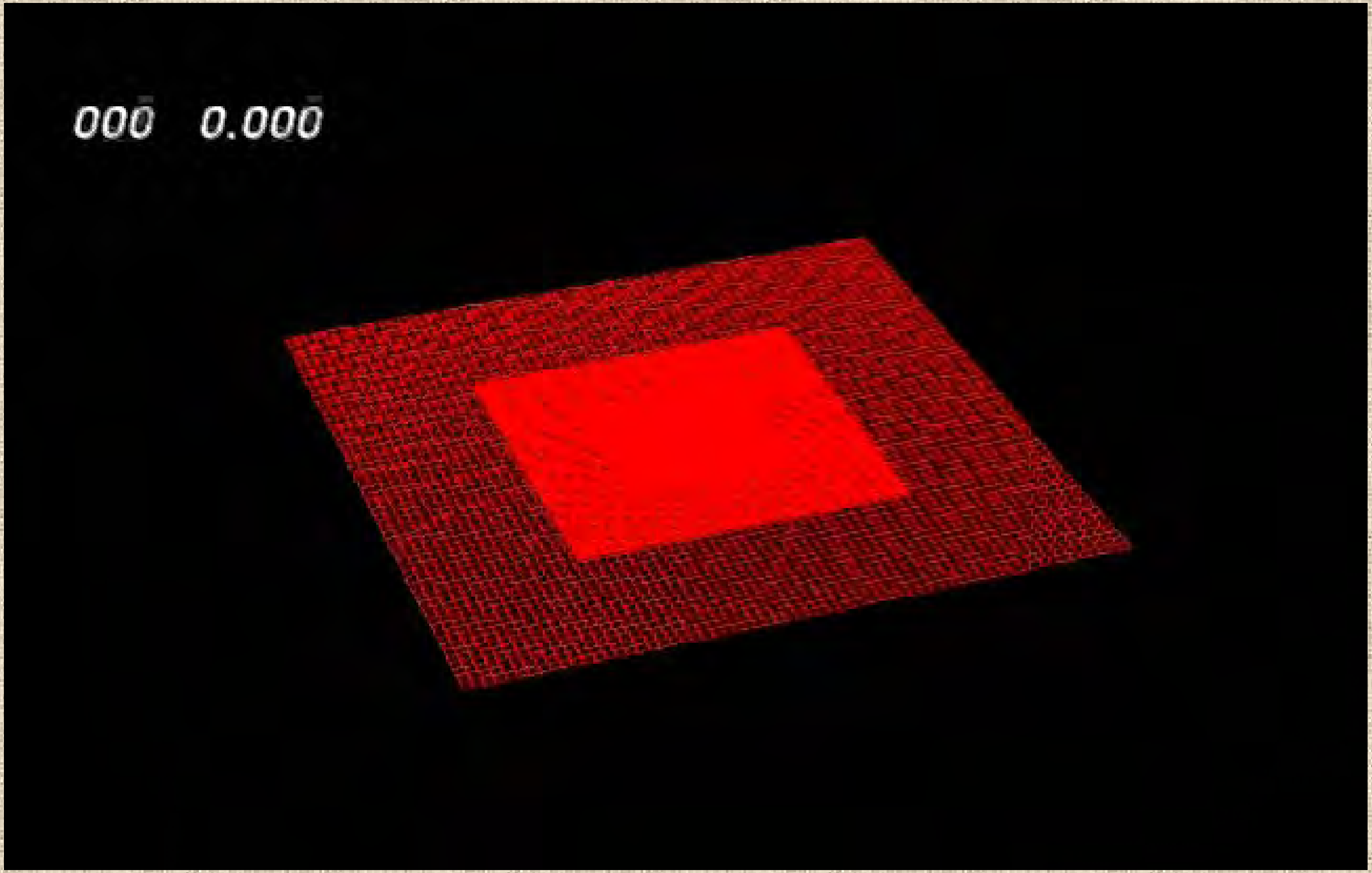
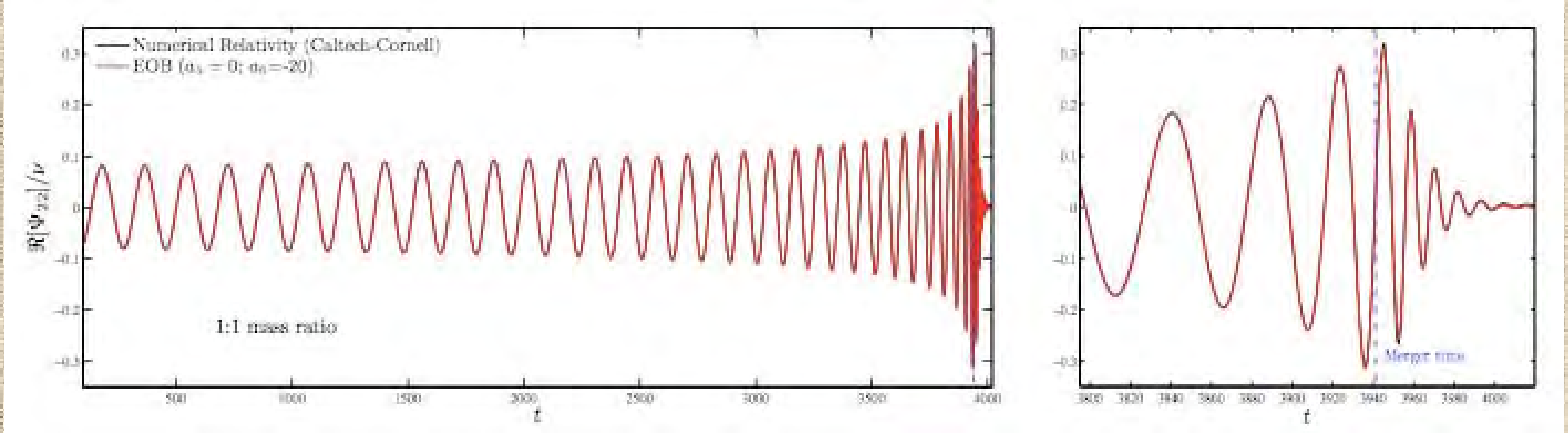


Grid structure and conformal factor

Equal mass, non-spinning holes: wave forms



Consistency across different codes with different formulations, and also with post-newtonian calculations (effective one-body problem, EOB).

Results are scale invariant.

Roughly 5% of initial energy and 30% of initial angular momentum are lost to gravitational waves!

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Gravitational recoils: kicks!

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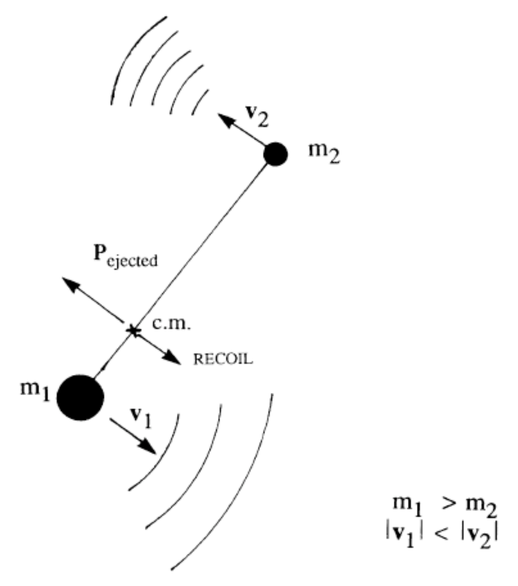


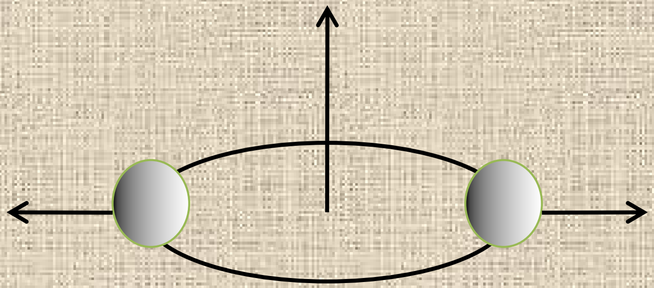
For BH’s with different spins and/or unequal masses, the emission of gravitational waves is asymmetric and they are “beamed” preferentially along a given direction.

The end result is that the waves carry away linear momentum, and the final merged black hole receives a kick.

For large enough kicks, the final black hole can reach velocities of a few 1000 km/s, large enough to escape the host galaxy!

The “super kick” (Jena, Brownsville-RIT)

For initial spins in the orbital plane and opposite to each other, a very large out-of-plane kick is found of ~2500 km/s.



Simulation:
Manuela Campanlli
Carlos Lousto
Yosef Zlochower

Visualization:
Hans-Peter Bischof

CCRG
RIT

Movie courtesy of M. Campanelli

Conclusions

- There have been attempts to solve the binary black hole problem numerically for just over 40 years.
- In the last few years the problem has been finally solved!
- Gravitational wave forms are now being produced and can be given to the observers to match to their data.
- The solution has required advances in a number of different areas:
  - ✓ Better understanding of the evolution equations.
  - ✓ Better understanding of slicing conditions.
  - ✓ Dynamical shift conditions to move the BH’s smoothly.
  - ✓ No factoring out of singularities analytically.
  - ✓ Complex parallel 3D codes, with mesh refinement and very high resolution.