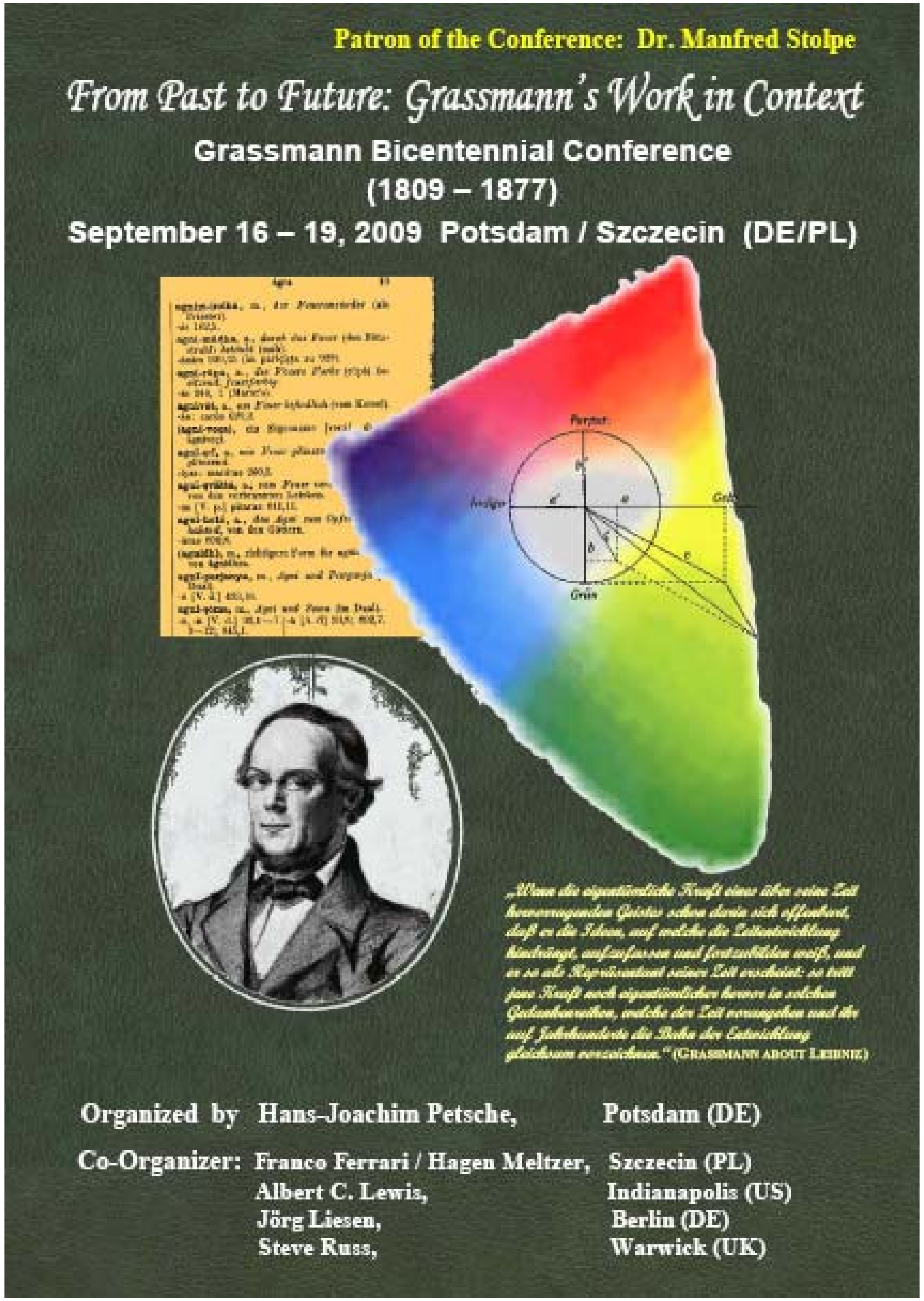
Hermann Graßmann and the Foundations of Linear Algebra

Jörg Liesen (TU Berlin)

SIAM Conference
on Applied Linear Algebra
Monterey, CA, USA
October 2009

Graßmann Bicentennial Conference



Leibniz envisions Linear Algebra in 1679

Gottfried Wilhelm Leibniz
(1646-1716),
German polymath

Christiaan Huygens
(1629-1695),
Dutch mathematician,
astronomer, physicist

Leibniz to Huygens on September 8, 1679:
“I am still not satisfied with algebra … I believe that, so far as geometry is concerned, we need still another analysis which is distinctly geometrical or linear and which will express *situation* directly as algebra expresses *magnitude* directly.”

More on Leibniz’ vision

Leibniz added:

“I believe that by this method one could treat mechanics almost like geometry, and one could even test the qualities of materials …

I have no hope that we can get very far in physics until we have found such a method …”

- Leibniz never discovered the “new method” he was envisioning.
- His letter to Huygens was published in 1833, and hence had little influence on the historical development.
- By the 1830s, others were already working on “Linear Algebra”.

Möbius' Barycentric Calculus (1827)

August Ferdinand Möbius
(1790-1868),
German mathematician
and astronomer

Der
barycentrische Calcul
ein neues Hülfsmittel
zur
analytischen Behandlung der Geometrie
dargestellt
und insbesondere
auf die Bildung neuer Classen von Aufgaben und
die Entwickelung mehrerer Eigenschaften der Kegelschnitte
angewendet
von
August Ferdinand Möbius

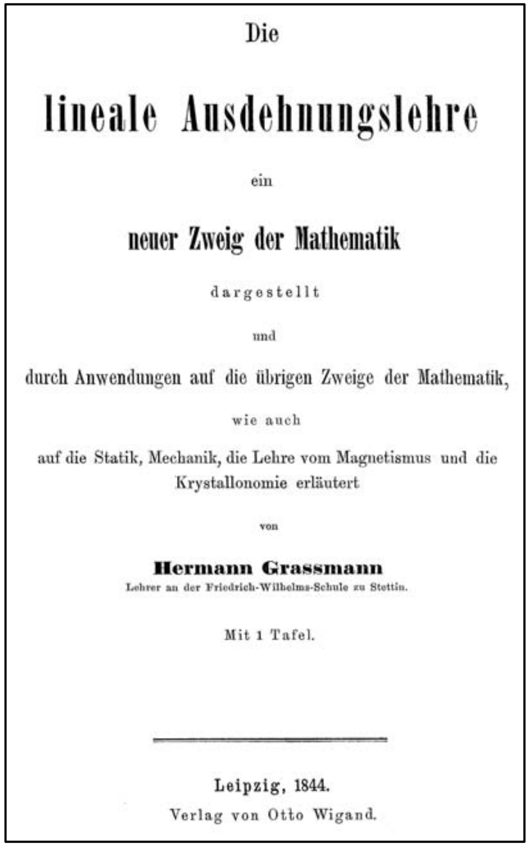
Leipzig
Verlag von Johann Ambrosius Barth
1827.

- One of the first systems in which geometrical entities (points) were operated on directly.
- Similar: Giusto Bellavitis' (1803-1880) "Calculus of Equipollences".
- Möbius showed how to add collinear line-segments, but gave no addition rules for non-collinear segements and no multiplication.
- Such a system was first constructed by Hermann Graßmann.

Hermann Günther Graßmann (1809-1877)

- Born April 15, 1809, in Stettin (Szczecin).
- His father Justus Graßmann, a gymnasium teacher, was the first to invent a purely geometrical product (“Raumlehre”, 1824).
- Hermann studied Theology and Philosophy in Berlin (1827-1830).
- Returned to Stettin to become a teacher, in the footsteps of his father.
- Never had any formal education in Mathematics.
- Was outside the mathematical establishement all his life.
- From the early 1830s he worked on restructuring the foundations of mathematics and he discovered “Linear Algebra”.
- His work “departed from all the then current mathematical traditions” (M. J. Crowe, A History of Vector Analysis, 1967).

The Ausdehnungslehre of 1844



Die

lineale Ausdehnungslehre

ein

neuer Zweig der Mathematik

dargestellt

und

durch Anwendungen auf die übrigen Zweige der Mathematik,

wie auch

auf die Statik, Mechanik, die Lehre vom Magnetismus und die Krystallonomie erläutert

von

Hermann Grassmann

Lehrer an der Friedrich-Wilhelms-Schule zu Stettin.

Mit 1 Tafel.

Leipzig, 1844.

Verlag von Otto Wigand.

- Graßmann's first major work: The Ausdehnungslehre (Theory of Extension) of 1844.
- Heavily influenced by philosophical considerations. A 12-page Preface ("Vorrede") and a 16-page Introduction ("Einleitung") explained his approach.

> "I had also realized that there must be a branch of mathematics which yields in a purely abstract way laws similar to those that in geometry seem bound to space.
> I soon realized that I had come upon the domain of a new science of which geometry itself is only a special application."

- His entirely abstract approach allowed him to consider new mathematical ideas such as *n*-dimensional spaces and noncommutative multiplication.
- The only comparable contemporary system: The quaternions of Sir William Rowan Hamilton (1805-1865), discovered on Oct. 16, 1843.

A flavour of Graßmann's writing

„Jede Strecke eines Systems m-ter Stufe kann als Summe von m Strecken, welche m gegebenen unabhängigen Aenderungsweisen des Systems angehören, dargestellt werden, aber auch jedesmal nur auf eine Art.“ (Ausdehnungslehre of 1844, § 20)

"Every displacement of a system of *m*-th order can be represented as a sum of *m* displacements belonging to *m* given independent methods of evolution of the system, the sum being unique for each such set."

(Translation: L. C. Kannenberg, 1995)

Reception of his contemporaries

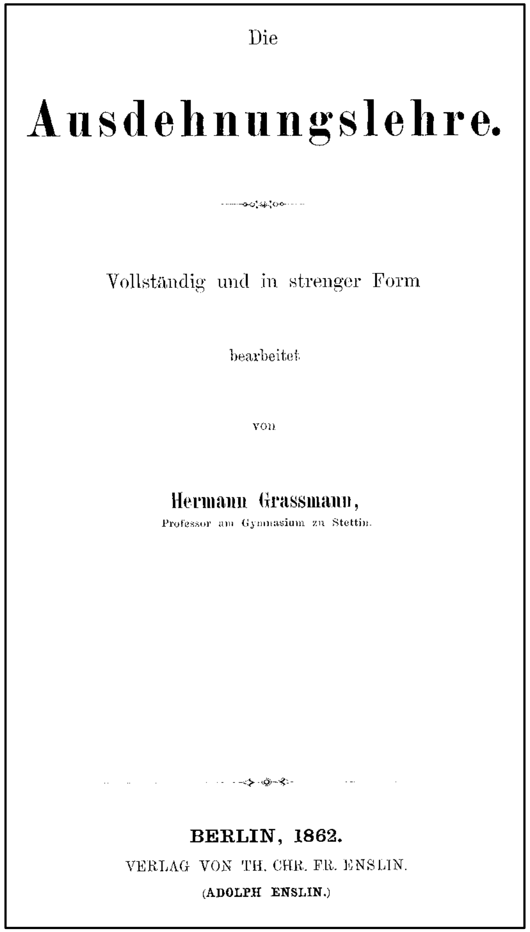
- The revolutionary content and the obscure style led to the "colossal neglect" (Crowe) of Graßmann's work.

Carl Friedrich Gauß (1777-1855) to Graßmann in Dec. 1844:
"… in order to discern the essential core of your work it is first necessary to become familiar with your special terminology. … however that will require of me a time free of other duties …"

Möbius to Ernst Friedrich Appelt (1812-1859) in Jan. 1846:
"… I have on numerous occasions attepmted to study it but I have never gone beyond the first sheets …"

Friedrich Engel (1861-1941) in Gesammelte Werke, Band 3:
"… the most painful experience for the author of a new work: his book nowhere received attention; the public was completely silent about it …".

The second Ausdehnungslehre (1862)


Die

Ausdehnungslehre.

Vollständig und in strenger Form

bearbeitet

von

Hermann Grassmann,

Professor am Gymnasium zu Stettin.

BERLIN, 1862.

VERLAG VON TH. CHR. FR. ENSLIN.

(ADOLPH ENSLIN.)

- Graßmann was convinced of his approach and decided to rewrite his work.
- He finished the second version in October 1861.
- 300 copies were printed at his own expense in the shop of his brother Robert.
- Written entirely in definition-theorem-proof style; almost no philosophical remarks.

Some of the Linear Algebra content in modern terms
(not all of these were firsts, but the completeness is striking):

theory of linear independence and dimension, exchange theorem, subspaces, join and meet of subspaces, projection of elements onto subspaces, change of coordinates, product structures including inner and outer multiplication, solution of systems of linear equations, orthogonality, linear transformations, matrix representation, rank-nullity theorem, eigenvalues and eigenspaces, characteristic polynomial, diagonalizability, primary decomposition theorem, law of inertia, applications to differential equations and real analysis …

(D. Fearnley-Sanders, Amer. Moth. Monthly, 86 (1979), gives a brief summary.)

Example: Orthogonal bases

152. Erklärung. Normal zu einander heissen zwei von null verschiedene Grössen, deren inneres Produkt null ist.

153. Erklärung. Normalsystem n-ter Stufe heisst ein Verein von n numerisch gleichen (von null verschiedenen) Grössen erster Stufe, von denen jede zu jeder normal ist;

154. Erklärung. Circuläre Aenderung nenne ich jede Transformation eines Vereins, durch welche 2 Grössen a und b des Vereins ſich beziehlich in xa + yb und in ∓(xb − ya) verwandeln, vorausgeſetzt, dass x² + y² = 1 ſei. Ich nenne die circuläre Aenderung eine poſitive oder negative, je nachdem a und b ſich in xa + yb und + (xb − ya), oder in xa + yb und − (xb − ya) verwandeln. Wenn hierbei x = cos. α und y = ſin. α ist, und a und b numerisch gleich und zu einander normal ſind, ſo ſage ich, der Verein habe ſich von a nach b hin um den Winkel α geändert.

Anm. Stellt man ſich unter a und b zwei gleichlange und zu einander ſenkrechte Strecken vor, ſo ſieht man leicht, dass durch die circuläre Aenderung, durch welche a in a₁ = a cos. α + b ſin. α, b in b₁ = b cos. α − a ſin. α übergeht, a₁ und b₁ von derſelben Länge ſind wie a und b und gegen einander ſenkrecht bleiben. Es bleiben

152. Two nonzero vectors are normal to one another if their inner product is zero.

153. A normal system of degree n is a collection of n mutually normal (nonzero) vectors of the same length.

154. A circular change of a collection of vectors is a transformation where two vectors, a and b, are respectively replaced by $xa+yb$ and $\pm(ya-xb)$, with $x^2+y^2=1$.

$$[a_1,\, b_1] = [a,\, b] \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

- Graßmann then analyzed the use of the method of circular change to transform a normal system into other systems (bases of the vector space).

(History of orthogonalization in a nutshell: Laplace, 1816; Gram, 1883; Schmidt, 1907.)

Example: Linear transformations

§. 4. **Ganze Funktionen ersten Grades. Quotient.**

377. Erklärung. Wenn $a_1, a_2, \cdots a_n$ Grössen erster (oder $n-1$-ter) Stufe in einem Hauptgebiet n-ter Stufe ſind, die in keiner Zahlbeziehung zu einander stehen, ſo verstehe ich unter dem Bruche (Quotienten)

$$Q = \frac{b_1, b_2, \cdots b_n}{a_1, a_2, \cdots a_n}$$

den Ausdruck, welcher mit $a_1, a_2, \cdots a_n$ multiplicirt, beziehlich die Werthe $b_1, b_2, \cdots b_n$ liefert, ſo dass alſo

$$\frac{b_1, b_2, \cdots}{a_1, a_2, \cdots} a_r = b_r.$$

Ich nenne $a_1, a_2, \cdots a_n$ die Nenner des Bruches, $b_1, b_2 \cdots b_n$ ſeine entsprechenden Zähler, und ſetze zwei Brüche, oder zwei Ausdrücke, welche aus Brüchen numerisch abgeleitet ſind, dann und nur dann einander gleich, wenn ſie mit jeder Grösse erster Stufe multiplicirt Gleiches liefern. Wenn auch die Zähler Grössen 1-ter oder $(n-1)$-ter Stufe ſind, und in keiner Zahlbeziehung zu einander stehen, ſo nenne ich den Bruch einen umkehrbaren, und bezeichne in dieſem Falle, wenn

$$Q = \frac{b_1, b_2, \cdots b_n}{a_1, a_2, \cdots a_n}$$

ist, mit $\frac{1}{Q}$ den umgekehrten Bruch, d. h. ich ſetze

$$\frac{1}{Q} = \frac{a_1, a_2, \cdots a_n}{b_1, b_2, \cdots b_n}.$$

- Graßmann defined a linear transformation by the image vectros for a given basis.
- He called this a “fraction” or “quotient” and denoted it by

$$Q = \frac{b_1, \ldots, b_n}{a_1, \ldots, a_n}, \qquad Q a_j := b_j, \quad j = 1, \ldots, n$$

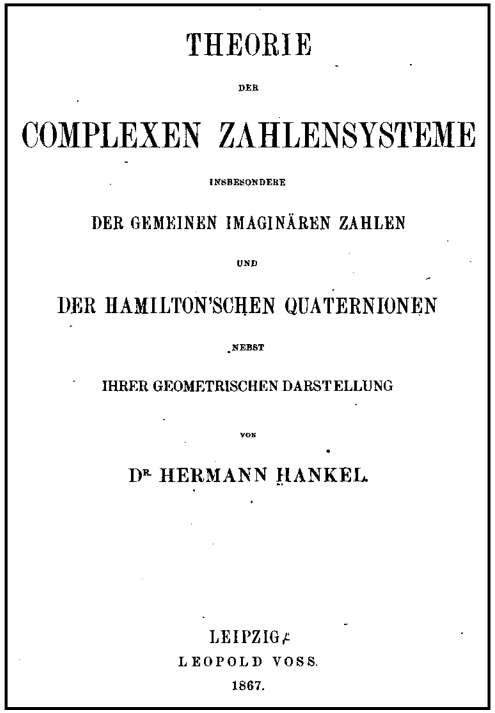
- Exchanging a basis vector, say $a_1 \to \hat{a}_1 = \sum_{j=1}^n \alpha_j a_j$, $\alpha_1 \neq 0$,

  so that $\hat{a}_1, a_2, \ldots, a_n$ still form a basis, yields

$$Q = \frac{b_1, b_2, \ldots, b_n}{a_1, a_2, \ldots, a_n} = \frac{\sum_{j=1}^n \alpha_j b_j, b_2, \ldots, b_n}{\hat{a}_1, a_2, \ldots, a_n}.$$

The gradual acceptance

- The 1862 Ausdehnungslehre initially was ignored, just as the 1844 version.
- Graßmann was disappointed, turned away from mathematics, and concentrated on other subjects, in particular linguistics.
- By the late 1860s, his work slowly was being recognized by other mathematicians.
- Among the first of them was Hermann Hankel (1839-1873), a student of Bernhard Riemann (1826-1866).



THEORIE

DER

COMPLEXEN ZAHLENSYSTEME

INSBESONDERE

DER GEMEINEN IMAGINÄREN ZAHLEN

UND

DER HAMILTON'SCHEN QUATERNIONEN

NEBST

IHRER GEOMETRISCHEN DARSTELLUNG

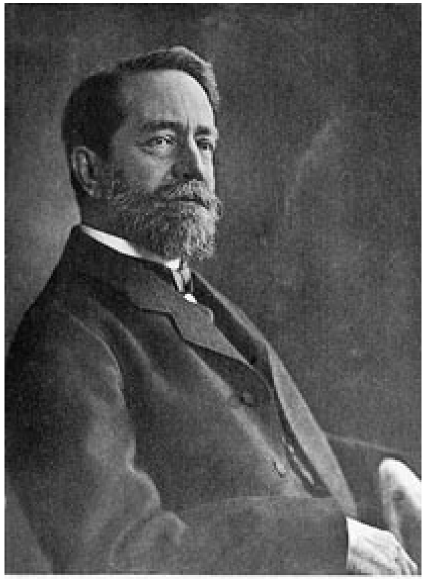
VON

Dr. HERMANN HANKEL.

LEIPZIG,
LEOPOLD VOSS.
1867.

"Roughly 10 percent of this book [of 1867] was devoted to Grassmann's system, on which Hankel bestowed much praise. Hankel's book was influential ..."
(M. J. Crowe, A History of Vector Analysis, 1967)

Major influences: Klein and Whitehead



- Felix Klein (1849-1925) learned about Graßmann from Hankel.
- He referred to Graßmann in his famous "Erlanger Programm" (1872).
- He played an important role in the publication of Graßmann's "Gesammelte Werke" (1894-1911).

Vergleichende Betrachtungen

über

neuere geometrische Forschungen

von

Dr. Felix Klein,

o. ö. Professor der Mathematik an der Universität Erlangen.

Programm

zum Eintritt in die philosophische Facultät und den Senat der k. Friedrich-Alexanders-Universität zu Erlangen.

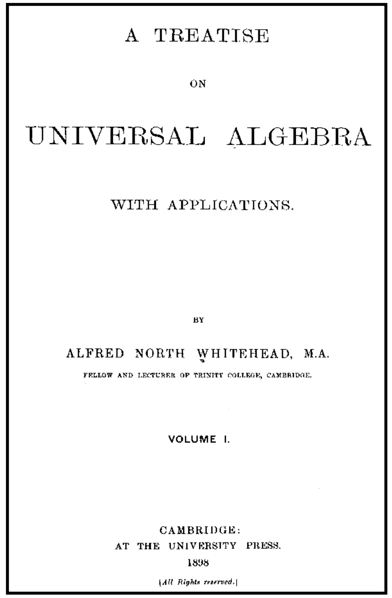
Erlangen.

Verlag von Andreas Deichert.

1872.

- Alfred North Whitehead (1861-1947) investigated systems of symbolic reasoning in his "Universal Algebra, Volume 1" (1898).
- Graßmann's calculus of extension was a chief example.

(There never was a Volume 2: After the International Congress in 1900, Whitehead and his student Bertrand Russel (1872-1970) started working on the "Principia Mathematica" (1902-1913).)



A TREATISE

ON

UNIVERSAL ALGEBRA

WITH APPLICATIONS.

BY

ALFRED NORTH WHITEHEAD, M.A.

FELLOW AND LECTURER OF TRINITY COLLEGE, CAMBRIDGE.

VOLUME I.

CAMBRIDGE:

AT THE UNIVERSITY PRESS.

1898

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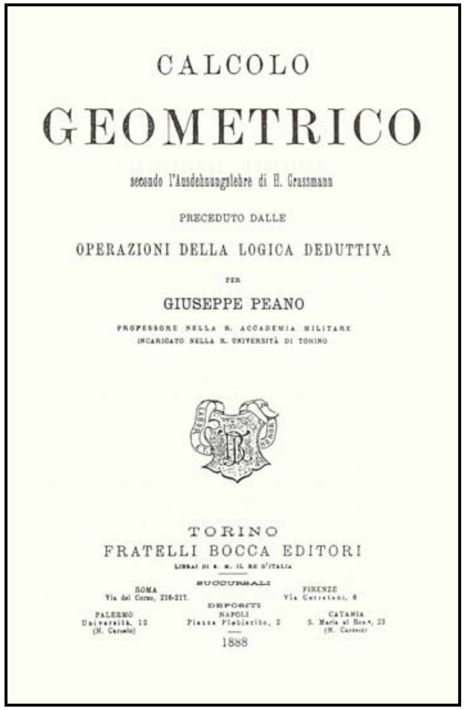
Major influences: Giuseppe Peano

- Giuseppe Peano (1858-1932) was deeply influenced by Graßmann.
- He gave his own version of Graßmann's work in his "Geometric Calculus according to the Ausdehnungslehre of H. Grassmann" (1888).
- The book contains the first axiomatic definition of a vector space.



CALCOLO
GEOMETRICO
secondo l'Ausdehnungslehre di H. Grassmann
PRECEDUTO DALLE
OPERAZIONI DELLA LOGICA DEDUTTIVA
PER
GIUSEPPE PEANO
PROFESSORE NELLA R. ACCADEMIA MILITARE
INCARICATO NELLA R. UNIVERSITÀ DI TORINO

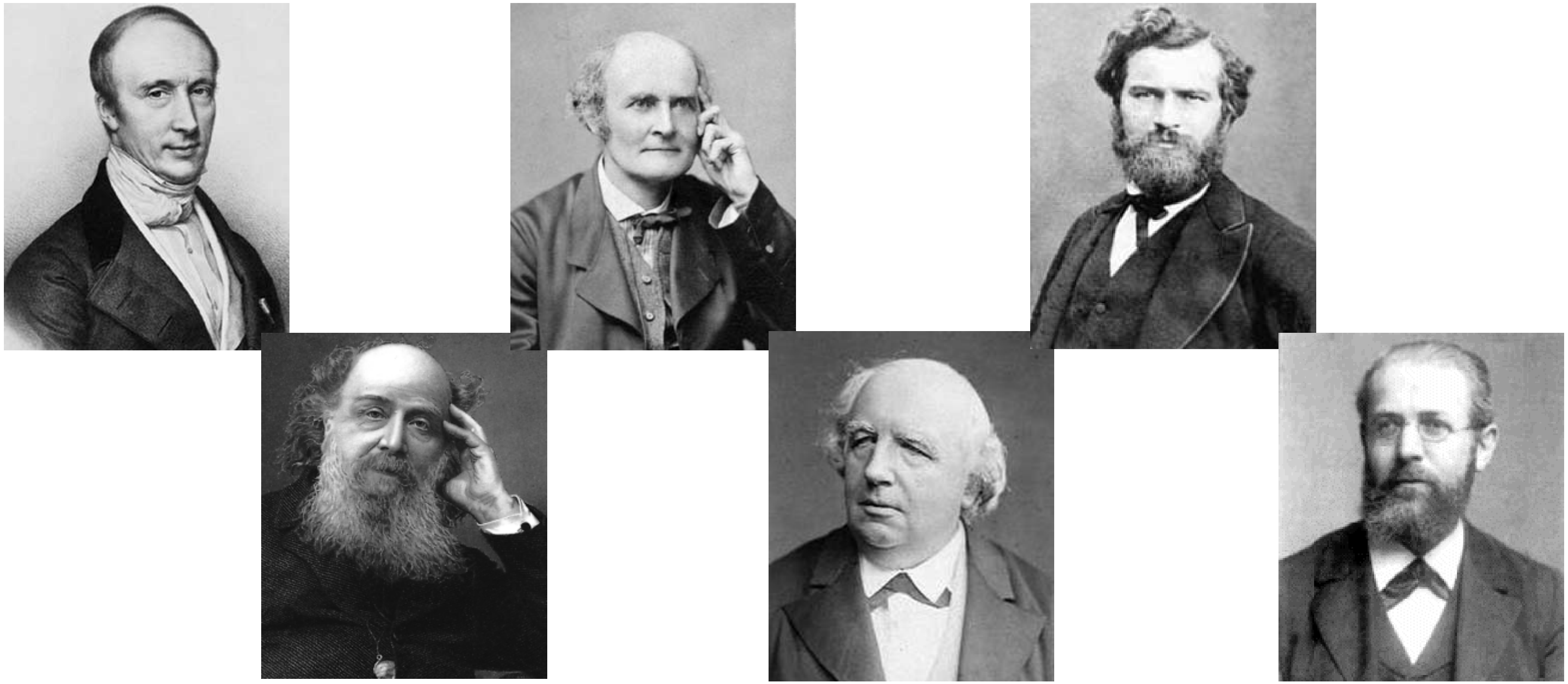
TORINO
FRATELLI BOCCA EDITORI
LIBRAI DI S. M. IL RE D'ITALIA
SUCCURSALI
ROMA Via del Corso, 216-217.
FIRENZE
DEPOSITI
NAPOLI Piazza Plebiscito, 2
1888

"A first attempt at a geometric calculus was due to the great mind of Leibniz (1679); in the present century there were proposed and developed various methods ... among which deserve special mention are the barycentric calculus of Möbius (1827), that of the equipollences of Bellavitis (1832), the quaternions of Hamilton (1852), and the applications to geometry of the Ausdehnungslehre of Hermann Grassmann (1844).
Of these methods, the last cited to a great extend incorporates the others and is superior in its powers of calculation and in the simplicity of its formulas."

(Translation: L. C. Kannenberg, 1997)

Major “non-influences”: Matrix Theory

- Graßmann had little to no impact in the development of matrix theory and in particular canonical forms.



Major “non-influences”: Relativity Theory

- Matrix and vector calculus became widely used after 1910, when they were successfully applied in Relativity Theory.
- Graßmann’s theory had little impact in that area.

Hermann Minkowski (1864-1909) in Dec. 1907:
“One may think that, instead of Cayley’s matrix calculus, Hamilton’s calculus of quaternions could be used, but it seems to me that the latter is too narrow and clumsy for our purpose.”

Edwin Bidwell Wilson (Bull. AMS, 21 (1915)):
“The analysis which is really appropriate to the theory of relativity as conceived by Minkowski is Grassmann’s. … Is it not unfortunate that Minkowski should have followed the English Cayley, referred to the Scot-Irish Hamilton, and ignored the Germann Grassmann? Should not some Geheimer Regierungsrat among his colleagues have given him secret directions to avoid such an unpatriotic scientific mésalliance?”

Concluding remarks

- The field of Linear Algebra developed historically in a very non-linear way because of the lack of unification:
  "The same author could use the same idea twice (in terms of the theory of linear algebra) in different contexts without noticing the similarity of the methods." (J.-L. Dorier, Historia Math., 22 (1995))

- Peano's axiomatic definition of a vector space (1888) remained largely unknown until Hermann Weyl stated it in the context of Relativity Theory in 1918 (referring only to Graßmann's "epoch making work").

- In the late 1920s axiomatization finally unified the field, which nowadays holds a central position in Mathematics.

- Many of Graßmann's Linear Algebra results were reestablished independently by others and now are associated with them (e.g. Steinitz' Exchange Thm.).

- His methods have been revived in the 20th century by the French school (Élie Cartan, Bourbaki), and his name today is attached to objects in Algebra (Grassmann algebra) and Differential Geometry (Grassmann manifold or Grassmanian).

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von

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Mit 1 Tafel.

Leipzig, 1844.

Verlag von Otto Wigand.

“One of the supreme landmarks in the history of the human mind.”
(George Sarton)

“A great classic in the history of mathematics, even though one of the most unreadable classics.”
(Michael J. Crowe)