The Early History of Matrix Iterations:

With a Focus on the Italian Contribution

SIAM Conference on Applied Linear Algebra
Monterey Bay – Seaside, California
26 October 2009

Michele Benzi
Department of Mathematics
and Computer Science
Emory University
Atlanta, GA
http://www.mathcs.emory.edu/~benzi

Outline

- Iterative methods prior to about 1930
- Mauro Picone and Italian applied mathematics in the Thirties
- Lamberto Cesari's work on iterative methods
- Gianfranco Cimmino and his method
- Cimmino's legacy

Iterative methods prior to about 1930

The earliest reference to an iterative approach to solving $A\mathbf{x} = \mathbf{b}$ appears to be contained in a letter by Gauss to his student Gerling dated 26 December 1823, in the context of solving least squares problems via the normal equations.

After briefly describing his method (essentially a relaxation procedure) on a 4×4 example, Gauss wrote:

You will in future hardly eliminate directly, at least not when you have more than two unknowns. The indirect procedure can be done while one is half asleep, or is thinking about other things.

Cf. *Werke, IX*, p. 278. See also E. Bodewig, *Matrix Calculus*, 1956.

Iterative methods prior to about 1930 (cont.)

In 1826 Gauss gave a block variant of the method in the *Supplementum* to his famous work on least squares, *Theoria Combinationis Observationum Erroribus Minimis Obnoxiae* (English translation by Pete Stewart published by SIAM in 1995).

Solution of normal equations by iteration became standard in 19th Century Germany, especially among geodesists and astronomers (including Gerling, Bessel, Schumacher,...).

According to Bodewig, Gauss had to solve systems with 20-30-40 unknowns. These systems were diagonally dominant and convergence was fast. In 1890, Nagel used iteration to solve a system of 159 unknowns arising in the triangulation of Saxony.

Iterative methods prior to about 1930 (cont.)

In 1845 Jacobi introduced his own iterative method, again for solving normal equations for least squares problems arising in astronomical calculations.[1]

In the same paper he makes use of carefully chosen plane rotations to increase the diagonal dominance of the coefficient matrix. This is perhaps the first occurrence of preprocessing of a linear system in order to speed up the convergence of an iterative method. He gives a 3×3 example.

Only in a subsequent paper (1846) he will use plane rotations to diagonalize a symmetric matrix.

[1] *Über eine neue Auflösungsart der bei der Methode der kleinsten Quadrate vorkommenden linearen Gleichungen*, Astronomische Nachrichten, 22 (1845), 297–306. Reprinted in *Gesammelte Werke*, vol. III, pp. 469–478.

Jacobi's example

Jacobi takes for his example a linear system that appears in Gauss' *Theoria Motus Corporum Coelestium in Sectionibus Conicis Solem Ambientium* (1809).

In modern notation, Jacobi wants to solve $A\mathbf{x} = \mathbf{b}$ where

$$A = \begin{bmatrix} 27 & 6 & 0 \\ 6 & 15 & 1 \\ 0 & 1 & 54 \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} 88 \\ 70 \\ 107 \end{bmatrix}.$$

Jacobi uses a plane rotation (with angle $\alpha = 22^{\circ}\,30'$) to annihilate the (1,2)-(2,1) coefficient. After this, the transformed system is solved in three iterations of Jacobi's method. Each iteration adds about one digit of accuracy.

Iterative methods prior to about 1930 (cont.)

Again in the context of least squares, in 1874 another German, Seidel, publishes his own iterative method. The paper contains what we now (inappropriately) call the Gauss–Seidel method, which he describes as an improvement over Jacobi's method.

Seidel notes that the unknowns do not have to be processed cyclically (in fact, he advises *against* it!); instead, one could choose to update at each step the unknown with the largest residual. He seems to be unaware that this is precisely Gauss' method.

In the same paper, Seidel mentions a block variant of his scheme. He also notes that the calculations can be computed to variable accuracy, using fewer decimals in the first iterations. His linear systems had up to 72 unknowns.

Iterative methods prior to about 1930 (cont.)

Another important 19th Century development that is worth mentioning were the independent proofs of convergence by Nekrasov (1885) and by Pizzetti (1887) of Seidel's method for systems of normal equations (more generally, SPD systems).

These authors were the first to note that a necessary and sufficient condition for the convergence of the method (for an arbitrary initial guess $\mathbf{x}^0$) is that all the eigenvalues of the iteration matrix must satisfy $|\lambda| < 1$. Nekrasov and Mehmke (1892) also gave examples to show that convergence can be slow.

Nekrasov seems to have been the first to relate the rate of convergence to the dominant eigenvalue of the iteration matrix. The treatment is still in terms of determinants, and no use is made of matrix notation.

Iterative methods prior to about 1930 (cont.)

In the early 20th century we note the following important contributions:

- The method of Richardson (1910);
- The method of Liebmann (1918).

These papers mark the first use of iterative methods in the solution of finite difference approximations to elliptic PDEs.

Richardson's method is still well known today, and can be regarded as an acceleration of Jacobi's method by means of over- or under-relaxation factors. Liebmann's method is identical with Seidel's.

Iterative methods prior to about 1930 (cont.)

The first systematic treatment of iterative methods for solving linear systems appeared in a paper by the famed applied mathematician Richard von Mises and his collaborator (and later wife) Hilda Pollaczek-Geiringer, titled *Praktische Verfharen der Gleichungsauflösung* (ZAMM **9**, 1929, pp. 58–77).

This was a rather influential paper. In it, the authors gave conditions for the convergence of Jacobi's and Seidel's methods, the notion of diagonal dominance playing a central role.

The paper also studied a stationary version of Richardson's method, which will be later called Mises' method by later authors.

Richard von Mises and Hilda Pollaczek-Geiringer

Two pioneers in the field of iterative methods.

Mauro Picone and the INAC

The paper by von Mises and Pollaczek-Geiringer was read and appreciated by Mauro Picone (1885-1977) and his collaborators at the *Istituto Nazionale per le Applicazioni del Calcolo* (INAC).

The INAC was among the first institutions in the world entirely devoted to the development of numerical analysis. Both basic research and applications were pursued by the INAC staff.

The Institute had been founded in 1927 by Picone, who was at that time professor of Infinitesimal Analysis at the University of Naples. In 1932 Picone moved to the University of Rome, and the INAC was transferred there. It became a CNR institute in 1933.

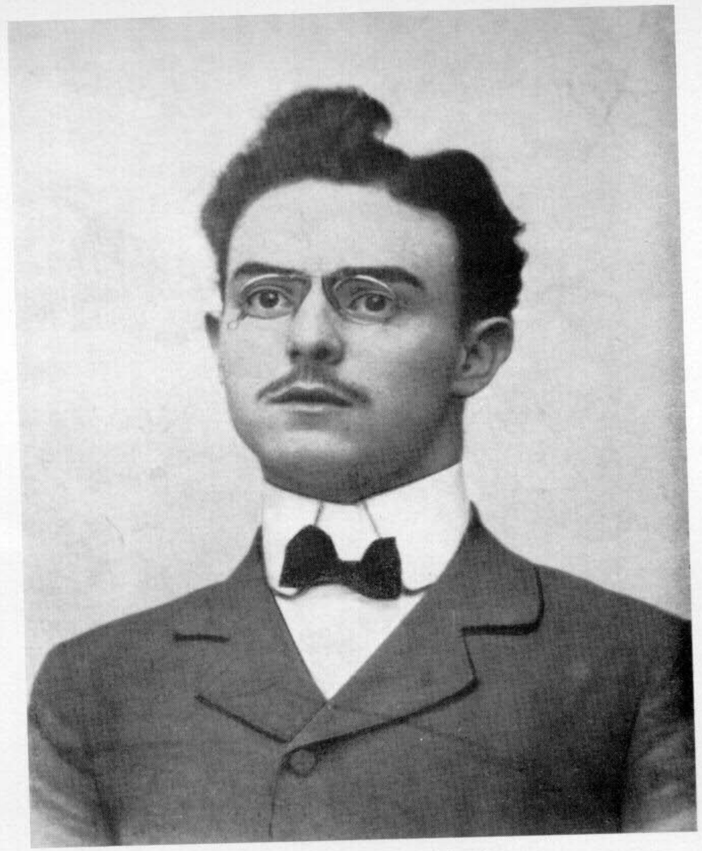
Mauro Picone and the INAC (cont.)

Picone's interest in numerical computing dated back to his service as an artillery officer in WWI, when he was put to work on ballistic tables.

The creation of the INAC required great perseverance and political skill on his part, since most of the mathematical establishment of the time was either indifferent or openly against it.

Picone's own training was in classical ("hard") analysis. He worked mostly on the theory of differential equations and in the calculus of variations. He was also keenly interested in constructive and computational functional-analytic methods.

Mauro Picone in 1903



Mauro Picone a 18 anni (nel 1903).

Mauro Picone around 1950

Mauro Picone and the INAC (cont.)

Throughout the 1930's and beyond, under Picone's direction, the INAC employed a number of young mathematicians, many of whom later became very well known.

INAC researchers did basic research and also worked on a large number of applied problems supplied by industry, government agencies, and the Italian military. Picone was fond of saying that

$$Matematica\ Applicata = Matematica\ Fascista$$

In the 1930s the INAC also employed up to eleven *computers* and *draftsmen*. These were highly skilled men and women who were responsible for carrying out all the necessary calculations using a variety of mechanical, electro-mechanical, and graphical devices.

Mauro Picone and the INAC (cont.)

As early as 1932, Picone designed and taught one of the first courses on numerical methods ever offered at an Italian university (the course was called *Calcoli Numerici e Grafici*). The course was taught in the School of Statistical and Actuarial Sciences, because Picone's colleagues in the Mathematics Institute denied his request to have the course listed among the electives for the degree in Mathematics.

The course covered root finding, maxima and minima, solutions of linear and nonlinear systems, interpolation, numerical quadrature, and practical Fourier analysis.

Both Jacobi's and Seidel's method are discussed (including block variants). Picone's course was not very different from current introductory classes in numerical analysis.

Mauro Picone and the INAC (cont.)

In the chapter on linear systems, Picone wrote:

The problem of solving linear systems has enormous importance in applied mathematics, since nearly all computational procedures lead to such problem. The problem can be considered as a multivariate generalization of ordinary division... While for ordinary division there exist automatic machines, things are not so for the solution of linear systems, although some great minds have set themselves this problem in the past. Indeed, the first researches go back to Lord Kelvin. Recently, Professor Mallock of Cambridge University has built a highly original electrical machine which can solve systems of $n \leq 10$ *equations in as many unknowns. This and other recent efforts, however, are far from giving a practical solution to the problem.*

Lamberto Cesari's work on iterative methods

Lamberto Cesari (Bologna, 1910; Ann Arbor, MI, 1990) studied at the *Scuola Normale* in Pisa under L. Tonelli, then in 1934 went to Germany to specialize under the famous mathematician C. Carathéodory. After his return from Germany, Cesari joined the INAC in Rome.

While at INAC Cesari wrote the paper *Sulla risoluzione dei sistemi di equazioni lineari per approssimazioni successive* (La Ricerca Scientifica, **8**, 1937, pp. 512–522.)

Cesari later became a leading expert in various branches of mathematical analysis and optimization. In 1948 he emigrated to the US, where he had a brilliant career, first at Purdue, then at Michigan.

Lamberto Cesari (1910-1990)

Lamberto Cesari's 1937 paper

Cesari gives a general theory of stationary iterations in terms of matrix splittings. Having written the linear system as

$$\omega A\mathbf{x} = \omega\mathbf{b}, \quad \omega \neq 0\,,$$

he introduces the splitting

$$\omega A = B + C, \quad \det(B) \neq 0\,,$$

and proves that the stationary iteration associated with this splitting converges for any choice of the initial guess if and only if the smallest (in magnitude) root of

$$\det(B + \lambda C) = 0$$

is greater than 1. This is equivalent to the usual condition $\rho(B^{-1}C) < 1$.

Lamberto Cesari's 1937 paper (cont.)

In the same paper, Cesari applies his general theory to the methods of Jacobi, Seidel, and von Mises (stationary Richardson). He uses $\omega \neq 1$ only for the latter.

In the case of von Mises' method (analyzed for the SPD case), Cesari notes that, regardless of ω, the rate of convergence of the method deteriorates as the ratio of the extreme eigenvalues of A increases. He writes:

In practice, we found that already for $\frac{\lambda_{\text{max}}(A)}{\lambda_{\text{min}}(A)} > 10$ *the method of von Mises converges too slowly.*

This observation leads Cesari to the idea of polynomial preconditioning.

Lamberto Cesari's 1937 paper (cont.)

Given estimates $a \approx \lambda_{\text{min}}(A)$ and $b \approx \lambda_{\text{max}}(A)$, Cesari determines the coefficients of the polynomial $p(x)$ of degree k such that the ratio of the maximum and the minimum of $q(x) = xp(x)$ is minimized over $[a, b]$, for $1 \leq k \leq 4$.

The transformed system

$$p(A)A\mathbf{x} = p(A)\mathbf{b}\,,$$

which he shows to be equivalent to the original one, can be expected to have a smaller condition number.

Cesari ends the paper with a brief discussion of when this approach may be useful and gives the results of numerical experiments with all three methods on a 3×3 example using a polynomial of degree $k = 1$.

Lamberto Cesari's 1937 paper (cont.)

Cesari's paper was not without influence: it is cited, sometimes at length, in important papers by Forsythe (1952-1953) and in the books by Bodewig (1956), Faddeev & Faddeeva (1960), Householder (1964), Wachspress (1966) and Saad (2003) among others.

It is, however, not cited in the influential books of Varga (1962) and Young (1971).

Cesari's paper is important for our story also because of the effect it had on a former student and assistant of Picone.

Cimmino's method

Gianfranco Cimmino (Naples, 1908; Bologna, 1989) graduated at Naples with Picone in 1927 with a thesis on approximate solution methods for the heat equation in 2D.

After a period spent at INAC and a study stay in Germany (again with Carathéodory), he undertook a brilliant academic career. He became a full professor in 1938, and in 1939 moved to the chair of Mathematical Analysis at Bologna, where he spent his entire career.

Cimmino's work was mostly in analysis: theory of linear elliptic PDEs, calculus of variations, integral equations, functional analysis, etc. He also wrote 5-6 short papers on matrix computations.

Gianfranco Cimmino (1908-1989)

Cimmino's method (cont.)

In 1938 *La Ricerca Scientifica* published a short (8 pages) paper by Cimmino, titled *Calcolo approssimato per le soluzioni dei sistemi lineari*.

As Picone himself explained in a brief introductory note, after reading Cesari's paper Cimmino reminded him of a method that he had developed around 1932 while he was at INAC. Cimmino's method did not appear to fit under Cesari's "systematic treatment," yet

it is most worthy of consideration in the applications because of its generality, its efficiency and, finally, because of its guaranteed convergence which can make the method practicable in many cases. Therefore, I consider it useful to publish in this journal Prof. Cimmino's note on the above mentioned method, note that he has accepted to write upon my insistent invitation.

Cimmino's method (cont.)

Cimmino considers the system $A\mathbf{x} = \mathbf{b}$ where A is a real $n \times n$ matrix, initially assumed to be nonsingular.

If $\mathbf{a}_i^T = [a_{i1}, a_{i2}, \ldots, a_{in}]$ denotes the ith row of A, the solution $\mathbf{x}_* = A^{-1}\mathbf{b}$ is the unique intersection point of the n hyperplanes described by

$$\langle \mathbf{a}_i, \mathbf{x} \rangle = b_i, \quad i = 1, 2, \ldots, n\,. \tag{1}$$

Given an initial approximation $\mathbf{x}^{(0)}$, Cimmino takes, for each $i = 1, 2, \ldots, n$, the reflection (mirror image) $\mathbf{x}_i^{(0)}$ of $\mathbf{x}^{(0)}$ with respect to the hyperplane (1):

$$\mathbf{x}_i^{(0)} = \mathbf{x}^{(0)} + 2\,\frac{b_i - \langle \mathbf{a}_i, \mathbf{x}^{(0)} \rangle}{\|\mathbf{a}_i\|^2}\,\mathbf{a}_i\,. \tag{2}$$

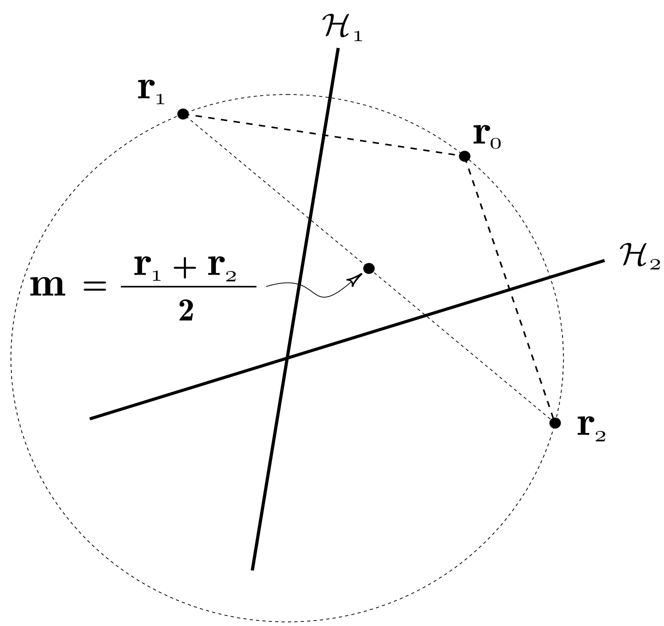
Cimmino's method (cont.)

Given n arbitrarily chosen positive quantities $m_1, \ldots, m_n$, Cimmino constructs the next iterate $\mathbf{x}^{(1)}$ as the center of gravity of the system formed by placing the n masses m_i at the points $\mathbf{x}_i^{(0)}$ given by (2), for $i = 1, 2, \ldots, n$. Cimmino notes that the initial point $\mathbf{x}^{(0)}$ and its reflections with respect to the n hyperplanes (1) all lie on a hypersphere the center of which is precisely the point common to the n hyperplanes, namely, the solution of the linear system. Because the center of gravity of the system of masses $\{m_i\}_{i=1}^n$ must necessarily fall inside this hypersphere, it follows that the new iterate $\mathbf{x}^{(1)}$ is a better approximation to the solution than $\mathbf{x}^{(0)}$:

$$\|\mathbf{x}^{(1)} - \mathbf{x}_*\| < \|\mathbf{x}^{(0)} - \mathbf{x}_*\| .$$

The procedure is then repeated starting with $\mathbf{x}^{(1)}$.

Cimmino's method ($n = 2$)



From C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, 2000.

Cimmino's method (cont.)

Cimmino proves that his method is always convergent.

In the same paper Cimmino shows that the iterates converge to a solution of $A\mathbf{x} = \mathbf{b}$ even in the case of a singular (but consistent) system, provided that rank$\,(A) \geq 2$.

He then notes that the sequence $\{\mathbf{x}^{(k)}\}$ converges even when the linear system is inconsistent, always provided that rank$\,(A) \geq 2$. Much later (1967) Cimmino wrote:

The latter observation, however, is just a curiosity, being obviously devoid of any practical usefulness. [sic!]

It can be shown that for an appropriate choice of the masses m_i, the sequence $\{\mathbf{x}^{(k)}\}$ converges to the minimum 2-norm solution of $\|\mathbf{b} - A\mathbf{x}\|_2 =$ min.

Cimmino's method (cont.)

In matrix form, Cimmino's method can be written as follows:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} + \frac{2}{\mu} A^T D(\mathbf{b} - A\mathbf{x}^{(k)})$$

($k = 0, 1, \ldots$), where

$$D = \text{diag}\left(\frac{m_1}{\|\mathbf{a}_1\|^2}, \frac{m_2}{\|\mathbf{a}_2\|^2}, \ldots, \frac{m_n}{\|\mathbf{a}_n\|^2}\right)$$

and $\mu = \sum_{i=1}^{n} m_i$.

Therefore, Cimmino's method is a special case of von Mises' method (stationary Richardson) on the normal equations if we let $m_i = \|\mathbf{a}_i\|^2$. Cimmino's method corresponds to using $\omega = 2/\mu$ for the relaxation factor. With such a choice, convergence is guaranteed.

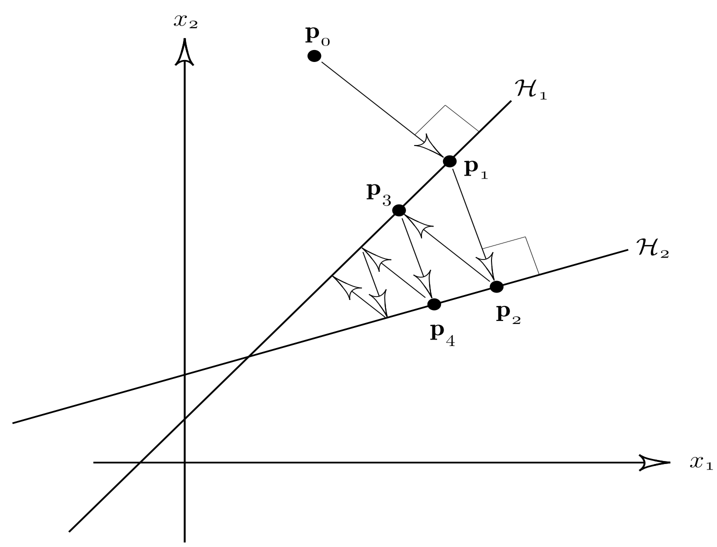
Cimmino's legacy

Cimmino's method, like the contemporary (and related) method of Kaczmarz, did not attract much attention until many years later.

Although it was described by Forsythe (1953) and in the books of Bodewig (1956), Householder (1964), Gastinel (1966) and others, I was able to find only 8 journal citations of Cimmino's 1938 paper until 1980.

After 1980, the number of papers and books citing Cimmino's (as well as Kaczmarz's) method picks up dramatically, and it is now in the hundreds. Moreover, both methods have been reinvented several times.

Kaczmarz's method ($n = 2$)



From C. D. Meyer, *Matrix Analysis and Applied Linear Algebra*, SIAM, 2000.

Cimmino's legacy (cont.)

Two major reasons for this surge in popularity are the fact that the method has the regularizing property when applied to discrete ill-posed problems, and the high degree of parallelism of the algorithm.

Today, Cimmino's method is rarely used to solve linear systems. Rather, it forms the basis for algorithms that are used to solve systems of inequalities (the so-called convex feasibility problem), and it has applications in computerized tomography, radiation treatment planning, medical imaging, etc.

Indeed, most citations occur in the medical physics literature, an outcome that would have pleased Gianfranco Cimmino.

References

M. Benzi, *Gianfranco Cimmino's contribution to numerical mathematics*, Atti del Seminario di Analisi Matematica dell'Università di Bologna, Technoprint, 2005, pp. 87–109.

Y. Saad and H. A. van der Vorst, *Iterative solution of linear systems in the 20th Century*, J. Comput. Applied. Math., 123 (2000), pp. 1–33.