Random matrices in Magnetic Resonance signal processing

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Abstract

Many important problems in the processing of Magnetic Resonance data can be reduced to well known ill-posed inverse problems. MR relaxometry, spectroscopy and MR image formation problems can be formulated respectively as the numerical inversion of Laplace transform, the modal analysis problem and the truncated trigonometric moment problem. A unified framework to efficiently solve these problems is provided by considering random Pade' approximants to the Z-transform $s(z)$ of the measured signal. It turns out that the singularities of $s(z)$ on the complex plane are the key quantities to make inference. Their location can be estimated from the logarithmic potential of the condensed density of the Pade' poles or, equivalently, of the eigenvalues of a random pencil of Hankel matrices. Several alternatives to compute these quantities are discussed and compared by numerical examples.

1 Introduction

Magnetic Resonance is a physical methodology based on the interaction between a system of atomic nuclei and an external magnetic field, allowing to obtain quantitative information about the internal structure of physical objects. First, the system of atomic nuclei in statistical equilibrium in the presence of a strong static magnetic field is excited by a radiofrequency pulse. After the pulse is switched off, a Free Induction Decay signal (FID) is recorded by an electrical coil surrounding the sample during the period when the system comes back to equilibrium. The NMR signal depends on some relevant physical quantities measuring the presence of atomic nuclei (spin density) and their magnetic interactions between themselves ($T2$ relaxation

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time) and with their chemical environment ($T1$ relaxation time).

Let us consider first the MR spectroscopy problem which consists in retrieving from a FID several physical parameters characterizing the sample under examination. It turns out that in most cases a good mathematical model for the FID is provided by the modal analysis model

$$a_k = s(t_k) + \nu_k,\ t_k = \Delta \cdot k,\ k = 0, \ldots, n-1,\ \ n \geq 2p,$$

where a_k is a sequence of equispaced data, ν_k is a white noise (gaussian) process and $s(t_k)$ are samples of the function

$$s(t; \underline{b}, \underline{z}) = \sum_{j=1}^{p} b_j {z_j}^t\ , \tag{1}$$

where $t \in \Re^+,\ \ \underline{z} = [z_1, \ldots, z_p]^T,\ \underline{b} = [b_1, \ldots, b_p]^T \in \mathbb{C}^p,\ \ 0 < |z_j| < 1,\ j = 1, \ldots, p.$ We want to estimate $p, \underline{b}, \underline{z}$ from the data a_k where

$$|\arg(z_j)| \leq \frac{\pi}{\Delta} \quad \forall j$$

in order to make the model identifiable from the data. Notice that we are not doing any hypothesis on the length of the interval spanned by the data in relation with the minimum separation between two consecutive $arg(z_j)$ therefore super-resolution problems are included in this setup. Usually fast Fourier transform based methods are used to estimate the unknown parameters. However severe problems arise, especially in "in-vivo" experiments, to cope with the decays estimation and, as a consequence, with the amplitude estimation too. In [2, 16] a new approach to solve this problem has been experimented in tumor markers investigations and studied in a deterministic framework [4]. - i.e. assuming that ν_k is a deterministic function mimicking a white noise process - and in a stochastic one [5].

Let us consider now the formation of MR images by standard FFT methods. The desired output is just the 2d Fourier transform of the data matrix. Due to physical constraints there is a symmetry break in the time required to get the data matrix. In fact the time required to add new rows to the this matrix is much larger than the time required to add new columns. The duration of the experiment is then essentially proportional to the number of rows. It is then necessary to reach a compromise between the number of rows - i.e. the number of Fourier coefficients in one direction - and the duration of the experiment. In many cases this results in the presence of the Gibbs oscillations in the estimated images due to the unsufficient number of measured Fourier coefficients with respect to the object characteristics. Of course if the signal-to-noise ratio (SNR) is too low the Gibbs artifact is masked by the noise. Therefore we are interested in the case when the SNR is high enough not to mask the Gibbs phenomenon but still low enough to cause problems with standard reconstruction methods.

The problem is stated as follows. Given a real interval $[-\pi, \pi]$ and $N+1$ numbers $-\pi \leq l_1 < l_2 \ldots < l_{N+1} \leq \pi$, let $\mathcal{F}$ be the class of functions defined as

$$F(t) = \sum_{j=1}^{N} w_j \chi_j(t) \ ,$$

where

$$\chi_j(t) = \begin{cases} 1 & \text{if } t \in [l_j, l_{j+1}] \\ 0 & \text{otherwise} \end{cases} \ ,$$

and the w_j are real weights. The problem consists in reconstructing a function $F(t) \in \mathcal{F}$ from a finite number of its noisy Fourier coefficients

$$a_k = \frac{1}{2} \int_{-\pi}^{\pi} F(t) e^{itk} dt + \nu_k = s_k + \nu_k \ , \quad k = 0, \ldots, D \ ,$$

where ν_k is a noisy perturbation. We are looking for a solution which is not affected by Gibbs artifact and can cope, stably, with the noise. The basic observation is the following. The unperturbed moments s_k are given by

$$s_k = \frac{1}{2} \int_{-\pi}^{\pi} F(t) e^{itk} dt = \sum_{j=1}^{N} w_j \frac{\sin(\beta_j k)}{k} \exp(i\lambda_j k) = \frac{1}{2i} \sum_{j=1}^{2N} v_j \frac{e^{i\alpha_j k}}{k} \ ,$$

where

$$\beta_j = \frac{l_{j+1} - l_j}{2}, \ \ \lambda_j = \frac{l_{j+1} + l_j}{2}, \ \ v_j = w_j, \ v_{N+j} = -w_j, \ \alpha_j = l_{j+1}, \ \alpha_{N+j} = l_j.$$

The reconstruction problem then reduces to the estimation of α_j and v_j, $j = 1, \ldots, N$, from a_k, $k = 0, \ldots, D$, which is a highly nonlinear problem. After the l_j are determined, the w_j can be computed by solving a linear problem. A new approach to solve this problem was experimented in [1] and studied in a deterministic framework in [3, 12, 13].

Finally let us consider the MR relaxometry problem which consists in estimating the density of the relaxation times $T1$. Data $a_k, k = 0, \ldots, n-1$ are measured such that

$$a_k = M_\infty + (m_0 - M_\infty) \int_0^\infty \rho(T1) e^{(-k \cdot \Delta / T1)} d\,T1 + \nu_k \tag{2}$$

where ν_k is a white noise (gaussian) process representing the measurement error. The problem consists of recovering the unknown constants m_0 and M_∞ and the function $\rho(T1)$ with the constraint $\rho(T1) \geq 0$ and $\int_0^\infty \rho(T1) d\,T1 = 1$. This problem is strongly related to the inversion of the Laplace transform, therefore it is an ill-posed problem. In fact, if $(M_\infty, \tilde{\rho}(t))$ solve the problem

$$\begin{cases} s_k = M_\infty + \int_0^\infty \tilde{\rho}(u) \exp(-ku) du \\ \tilde{\rho}(u) \geq 0 \end{cases} \tag{3}$$

then $(m_0, \rho(T1))$ given by

$$\rho(T1) = \frac{\tilde{\rho}(1/T1)}{T1^2} \cdot \frac{1}{\int_0^\infty \tilde{\rho}(u) du}$$

$$m_0 = M_\infty + \int_0^\infty \tilde{\rho}(u)du.$$

solve the problem

$$\begin{cases} s_k = M_\infty + (m_0 - M_\infty)\int_0^\infty \rho(T1)e^{(-k/T1)}d\,T1 \\ \rho(T1) \geq 0 \end{cases}$$

Moreover, by the transformation $\tilde{\rho}(u) = F(e^{-u})e^{-u}$ in (3), problem (2) can be transformed in

$$a_k = M_\infty + \int_0^1 F(t)t^k dt + \nu_k,\ F(t) \geq 0,$$

which is a constrained Hausdorff noisy moment problem. This problem was afforded in [6] where a general procedure is proposed partially based on the ideas shortly discussed in the next section and studied in [8].

2 Pade' approximants and the condensed density of poles

We show now how the modal analysis problem, the trigonometric moment problem and the inversion of Laplace transform of positive functions can be formulated and solved in a unified mathematical framework. A review for the first two problems is given in [7].
We first consider the Z-transform

$$s(z) = \sum_{k=0}^{\infty} s_k z^{-k}$$

of the ideal data $\{s_k\}$ which in the considered cases is given respectively by

$$s(z) = \sum_{j=1}^{p} \frac{b_j}{1 - z_j z^{-1}}\ ,$$

$$s(z) = \sum_{j=1}^{N} w_j \Big(\beta_j + \frac{1}{2i}\ln\frac{z - e^{il_j}}{z - e^{il_{j+1}}}\Big)$$

and, assuming for simplicity to know M_∞,

$$s(z) = \int_0^1 \frac{zF(t)}{z - t}dt$$

Then we consider Pade' approximants $[l, n]_s$ of orders $l < n$ to $s(z)$ that is:

$$[l, n]_s = P_1(z^{-1})/P_2(z^{-1})$$

$P_1(z)$, $P_2(z)$ polynomials of degree l and n, respectively such that

$$P_1(z^{-1}) - P_2(z^{-1})s(z) = O(z^{-(l+n+1)})$$

as $|z| \to \infty$. We then notice that some of the poles of $[l,n]_s$ must cluster around the singularities of $s(z)$. In fact this is the only way that the Pade' approximants which are continuous functions except at poles can approximate functions with singularities. But knowledge of the singularities of $s(z)$ in the first two cases solves the hardest part of the problem (the estimation of the non linear quantities z_j and l_j). In [12, 13, 3, 4] the asymptotic behavior of the poles of $[l,n]_f$, which are the only quantities computable from the measured data $\{a_k\}$, is studied and their connection with the singularities of $s(z)$ is explicited.

To solve the last problem we need to invert the Stieltjes transform of $s(z)$ with the non-negativity constraint. This can be done by using the following potential theory result :

$$F(t) \propto e^{\int \log \frac{1}{|x-t|} h(x)dx}$$

$h(x)$: asymptotic spectral density of the Pade' poles $\{z_j^{(n)}\}$i.e.

$$h(x) = \lim_{n\to\infty} \rho_n(x)$$

$$\rho_n(x) = \frac{1}{n}\sum_{j=1}^{n} \delta(x - z_j^{(n)}),$$

where δ denotes the Dirac's measure. Hence $\log F(t)$ is the restriction to the real line of the solution of the Poisson equation

$$\Delta \log F(z) = h(z).$$

In ([8]) is shown how $h(z)$ can be computed starting from the poles of $[l,n]_f$ which in this case are assumed to be random functions. In fact this is the proper framework where all of the above stated problems should be solved. We can only compute the Pade' approximant $[l,n]_f$ to

$$f(z) = \sum_{k=0}^{\infty} a_k z^{-k} \ . \tag{4}$$

which is a random function as a_k are random variables. Therefore also $[l,n]_f$ will be random functions as well as their poles. The key quantity to make inference according the procedures outlined above is then the condensed density function of the poles of $[l,n]_f$ defined as

$$h_n(z) = E[\rho_n(z)] = \frac{1}{n} E\left[\sum_{j=1}^{n} \delta(z - z_j)\right]. \tag{5}$$

In fact clusters of poles of $[l,n]_s$ will correspond to regions A where the relative density of poles is large i.e. $\int_A h_n(z)dz$ is large. It turns out that $h_n(z)$ is also the marginal density of each pole z_j.We then make the assumption that the SNR is large enough to be able to make inference about the singularities of $s(z)$ using the information provided by $h_n(z)$. In fact as SNR becomes smaller $h_n(z)$ becomes closer to the condensed density of the poles of the Pade' approximants to the $Z-$transform of a white noise which is concentrated around the unit circle ([9]), providing therefore no information about $s(z)$.

3 Condensed density and pseudospectra

We now discuss several methods for computing $h_n(z)$. First we note that the poles of $[l,n]_f$ are the roots of the Hadamard polynomials $H_n^{(l)}(z) = z^n P_2(z^{-1})$ which satisfy the following generalized orthogonality relations:

$$\int_\Gamma H_n^{(l)}(z) f(z) z^{l-n+k} dz = 0,\ k = 0, \cdots, n-1$$

where Γ is a positively oriented Jordan curve enclosing the singularities of $f(z)$. It turns out that if we consider the random matrices

$$U^{(q)} = \begin{pmatrix} a_q & a_{q+1} & \cdots & a_{q+n-1} \\ a_{q+1} & a_{q+2} & \cdots & a_{q+n} \\ . & . & . & . \\ a_{q+n-1} & a_{q+n} & \cdots & a_{q+2n-2} \end{pmatrix}$$

the roots $\{z_j\}$ of $H_n^{(l)}(z)$ are the generalized eigenvalues of the Hankel pencil

$$U^{(q+1)} - zU^{(q)},\ q = l - n + 1.$$

A straightforward way of computing $h_n(z)$ is the following. From the definition of condensed density (5) we have to compute the average number of poles falling in each infinitesimal region of the complex plane. For a given realization of the data process $\{a_k\}$ we can then compute the generalized eigenvalues and their empirical distribution. If n is small this turns out to be a poor approximation. By exploiting the fact that for each order $[l,n]_f$ are approximating the same function we can can pool all the poles of $[l,p]_f$, $\forall p \leq n$ and consider again their empirical distribution. This second method turns out to be reliable for small n too. Most of the applications on MR quoted above are based on this estimator. We introduce now two others less empirical estimators of $h_n(z)$. By using the classic result

$$\Delta \log \frac{1}{|z|} = -2\pi\delta(z)$$

where Δ denotes the Laplacian operator, it can be shown ([9]) that the expression (5) can be rewritten as

$$h_n(z) = \frac{1}{4\pi n} \Delta \left[E \left\{ \log \left(\left| det(U^{(q+1)} - zU^{(q)}) \right|^2 \right) \right\} \right],$$

where the expectation is now with respect to the joint distribution of the data $\{a_k\}$, which is a multivariate Gaussian distribution with mean $\{s_k\}$ and covariance $\sigma^2 I_n$. Let be $\hat{a}_k$ the measured realization of the data process $\{a_k\}$ and assume that the noise variance σ^2 is known. We can consider $\hat{a}_k$ as an approximation of the unknown signal s_k. Given a sample ϵ_k from $N(0, \sigma^2)$, the random variable $\tilde{a}_k = \hat{a}_k + \epsilon_k$, distributed as $N(s_k, \sigma^2)$ can be used in a MonteCarlo integration method. In fact $h_n(z)$ can be approximated by computing

$$\hat{h}_n(z) = \frac{1}{4\pi n N} \tilde{\Delta} \left[\sum_{j=1}^{N} \left\{ \log \left(\left| det(U_j^{(q+1)} - zU_j^{(q)}) \right|^2 \right) \right\} \right],$$

where $\tilde{\Delta}$ is a discrete approximation of the Laplacian, $\tilde{a}_k^{(j)}, j = 1\ldots, N$ are independent realizations of $\tilde{a}_k$ and by $U_j^{(q)}$ we denote the Hankel matrices $U^{(q)}$ whose elements are $\tilde{a}_k^{(j)}$. In order to compute $det(U_j^{(q+1)} - zU_j^{(q)})$ we notice that $U_j^{(q)}$ is invertible with probability one therefore

$$det(U_j^{(q+1)} - zU_j^{(q)}) = det(U_j^{(q)}) \cdot det[(U_j^{(q)})^{-1}U_j^{(q+1)} - zI].$$

If we denote by $\Lambda^{(j)}$ the spectrum of $(U_j^{(q)})^{-1}U_j^{(q+1)}$ we have

$$\log\left(\left|det(U_j^{(q+1)} - zU_j^{(q)})\right|^2\right) = 2\log(|det(U_j^{(q)})|) + 2\sum_{\lambda\in\Lambda^{(j)}} \log|\lambda + z|.$$

Therefore we do not need to solve a generalized eigenvalue problem for each z in a suitable domain of the complex plane but it is enough to compute $\Lambda^{(j)}, j = 1, \ldots, N$ and then use the above formula.
In order to avoid MonteCarlo integration we now introduce a different method to approximate the condensed density based on the pseudospectrum of the generalized eigenvalues which can be computed by fast algorithms even when n is very large ([15, 17]). Let us recall that given a complex matrix A and a real number $\epsilon > 0$ the $\epsilon-$pseudospectrum of A is defined by

$$\Lambda_\epsilon = \{z \in \mathbb{C} : z \in \Lambda(A + E) \text{ for some } E \text{ with } \|E\| \leq \epsilon\}$$

or, equivalently, by

$$\Lambda_\epsilon = \{z \in \mathbb{C} : \sigma_{min}(A - zI) \leq \epsilon\}$$

where $\Lambda(A)$ is the set of eigenvalues of A and $\sigma_{min}(A)$ is the smallest singular value of A. In the following we will denote by Λ_ϵ the $\epsilon-$pseudospectrum of $(U^{(q)})^{-1}U^{(q+1)}$ and by Λ its spectrum. Let us define the set

$$H_\delta = \{z \in \mathbb{C} : h_n(z) > \delta\}.$$

Let us make the following

Assumption 1. *$h_n(z) \in C^{(0)}(D)$, where D is a compact set including the unit circle and its local maxima in D correspond to points $\lambda \in \Lambda$.*

We want to show that

Proposition 1. *If the assumption is true, $\exists\ \delta^*$ such that $\forall\delta < \delta^*$ there exist $0 < \epsilon_3 < \epsilon_4$ dependent on δ such that $\Lambda_{\epsilon_3} \subset H_\delta \subset \Lambda_{\epsilon_4}$, while for $\delta \geq \delta^*$, $H_\delta \subset \Lambda_{\epsilon_4}$.*

Let us prove first the following lemmas

Lemma 2. *$\forall\delta \in (0, \max_z h_n(z))$, there exist $\epsilon_1 > 0, \epsilon_2 > 0$ dependent on δ and $\Lambda^* \subseteq \Lambda$ such that $\bigcup_{\lambda\in\Lambda^*} N_{\epsilon_1}(\lambda) \subset H_\delta \subset \bigcup_{\lambda\in\Lambda^*} N_{\epsilon_2}(\lambda)$ where $N_\epsilon(\lambda) = \{z \in \mathbb{C} : |z - \lambda| \leq \epsilon\}$*

Proof. Given $\delta \in (0, \max_z h_n(z))$, let us decompose the set H_δ into connected sets $H_\delta^{(k)}$ i.e. $H_\delta = \bigcup_k H_\delta^{(k)}$. By Assumption 1, each $H_\delta^{(k)}$ will contain at least one $\lambda_k \in \Lambda$ in its interior. Let us define $\Lambda^* = \{\lambda \in \Lambda, \lambda \in H_\delta, \lambda \notin \partial H_\delta^{(k)}, \forall k\}$. For all $\lambda_k \in \Lambda^*$ let us define $\eta_k = min_{z \in \partial H_\delta^{(k)}} |z - \lambda_k|$. By the continuity of $h_n(z)$, it follows that $\eta_k > 0$. We then put $\epsilon_1 = \min_k \eta_k$. It follows that $\emptyset \neq \bigcup_{\lambda \in \Lambda^*} N_{\epsilon_1}(\lambda) \subset H_\delta$. If we now define $\epsilon_2 = \max_k \max_{z \in \partial H_\delta^{(k)}} |z - \lambda_k|$, we get $H_\delta \subset \bigcup_{\lambda \in \Lambda^*} N_{\epsilon_2}(\lambda)$. □

Lemma 3. *If $\lambda \in \Lambda$ then $N_\epsilon(\lambda) \subset \Lambda_\epsilon$. Conversely if $z \in \Lambda_\epsilon$ then $\exists\ \lambda \in \Lambda$ and $\delta_\epsilon = C \cdot \epsilon$ such that $|z - \lambda| \leq \delta_\epsilon + O(\epsilon^2)$.*

Proof. If $\lambda \in \Lambda \Rightarrow \exists\ y$ such that $(U^{(q)})^{-1} U^{(q+1)} y = \lambda y$. If $|z - \lambda| \leq \epsilon \Rightarrow \exists\ \eta,\ |\eta| \leq \epsilon$ such that $z = \lambda + \eta$. But then if $E = \eta I$ we get $\|E\| = |\eta| \leq \epsilon\ \forall$ norm such that $\|I\| = 1$ and $(U^{(q)})^{-1} U^{(q+1)} + E)y = \lambda y + \eta y = zy$. Conversely let be $A = (U^{(q)})^{-1} U^{(q+1)} + E$ and let us consider the perturbation $(U^{(q)})^{-1} U^{(q+1)} = A - E$ with $\|E\| \leq \epsilon$. From [14] if $z \in \Lambda(A)\ \exists\ \lambda \in \Lambda$ such that $|z - \lambda| \leq \epsilon \|y\|_2 + O(\epsilon^2)$ where y is a left eigenvector corresponding to z. Hence the proof is completed by posing $\delta_\epsilon = \epsilon \|y\|_2$. □

Proof. (**of Proposition 1.**) We have that $\bigcup_{\lambda \in \Lambda^*} N_{\epsilon_2}(\lambda) \subset \Lambda_{\epsilon_2}$. In fact if $z \in \bigcup_{\lambda \in \Lambda^*} N_{\epsilon_2}(\lambda)$ then $z \in N_{\epsilon_2}(\lambda)$ for some $\lambda \in \Lambda^*$. But then $\forall \delta$ from the first part of Lemma 3 it is sufficient to choose $\epsilon_4 = \epsilon_2$ to prove that $H_\delta \subset \Lambda_{\epsilon_4}$. To finish the proof we notice that it must exist a δ^* such that $\forall \delta < \delta^*,\ \Lambda^* = \Lambda$. In this case, given $\epsilon_3 > 0$, $\forall z \in \Lambda_{\epsilon_3}$, from the second part of Lemma 3 it must exist $\lambda \in \Lambda^*$ such that $|z - \lambda| \leq C \cdot \epsilon_3 + O(\epsilon_3^2)$. By choosing ϵ_3 small enough to get $\delta_{\epsilon_3} < \epsilon_1 - O(\epsilon_3^2)$ we have $\Lambda_{\epsilon_3} \subset \bigcup_{\lambda \in \Lambda} N_{\epsilon_1}(\lambda)$. The conclusion follows by Lemma 2. □

4 Computation of the pseudospectrum

From Proposition 1 it follows that we can approximate the level sets of $h_n(z)$ by the $\epsilon-$pseudospectrum of $(U^{(q)})^{-1} U^{(q+1)}$ at least for $\delta < \delta^*$. We show now how Λ_ϵ can be computed quickly and stably. Following [11], we first note that as $U^{(q)})$ can be ill-conditioned it is convenient not to form its inverse explicitly. Instead the generalized Schur decomposition of the pencil can be considered:

$$U^{(q)} = QSZ,\ \ U^{(q+1)} = QTZ,\ Q^H Q = Z^H Z = I$$

and S and T are upper triangular matrices. By using these transformations we get

$$(U^{(q)})^{-1} U^{(q+1)} - zI = Z(S^{-1}T - zI)Z^H$$

hence

$$\Lambda_\epsilon = \{z \in \mathbb{C} : \sigma_{min}\left((U^{(q)})^{-1} U^{(q+1)} - zI\right) = \sigma_{min}(L - zI) \leq \epsilon\},$$

where $L = (S^{-1}T)^H$ is a lower triangular matrix. The generalized Schur decomposition is usually performed by the QZ algorithm that first reduces the matrices in the pencil respectively to a triangular and an upper Hessenberg matrices by orthogonal transformations and then transforms them to triangular matrices by a deflation procedure. By exploiting the replication of columns in the matrices $U^{(q)}$) and $U^{(q+1)}$ in [11] the authors suggest to make the Hessemberg-triangular reduction step by performing a QR factorization of the augmented matrix $[U^{(q)}|U^{(q+1)}e_n]$ where e_n indicates the last column of the identity matrix of order n. This can be performed in $4/3n^3$ flops instead than $8n^3$ flops. Up to this point all the computations must to be done only once. Now we have to compute $\forall z$ in a suitable grid in the complex plane the smallest singular value of the matrix $L - zI$. This can be done by several methods such as e.g. the one described in [10] which is specially suited to compute the smallest singular value σ_{min} and provides lower bounds on it that can be used to pruning regions where $\sigma_{min} > \epsilon$. However in the following we used the method suggested in [15] which includes special devices to speed up the computation of the pseudospectrum.

5 Experimental results

In order to illustrate the behavior of the different estimators of the condensed density illustrated above, a noisy sample of the sinc function was considered. More specifically

$$a_k = \frac{\sin k}{k} + \nu_k,\ k = 0, \ldots, 99$$

and ν_k is a sample from a Gaussian zero mean distribution with standard deviation $\sigma = 0.0001$. The asymptotic behavior of the poles of $[l, n]_f$ in this case have been described in [13] and it is characterized by clusters of poles around a curve C connecting the branch points of $s(z)$ which are $e^{\pm i}$, and clusters of poles around the unit circle. The poles density along C tends to infinity approaching the branch points. Moreover two gaps with no poles are present at each side of the branch points along the unit circle.
In Fig.1 the estimation of $h_n(z)$ obtained by computing the empirical distribution of the pooled poles of the Pade' approximants for $n = 10, \ldots, 99$ is shown. In Fig.2 the estimation of $h_n(z)$ obtained by MonteCarlo integration is shown. Finally in Fig. 3 the pseudospectrum of the generalized eigenvalues is displayed. In all cases the clustering of poles along the expected curves and the gaps are recognizable. However if we plot level lines corresponding to values close to the absolute maxima of the condensed density, only the regions around the branch points emerge in the empirical and MonteCarlo estimators. This reflects the fact that the branch points are strong attractors of poles. However they disappear in the pseudospectrum, in agreement with Proposition 1, therefore this feature is not captured by the pseudospectrum. We notice also that the empirical distribution method is the fastest but the quality is poor, the MonteCarlo method provides the most clear picture but it costs 10 times more than the empirical distribution method. The pseudospectrum is a compromise between the two both in computational time (5 times more than the empirical distribution) and quality.

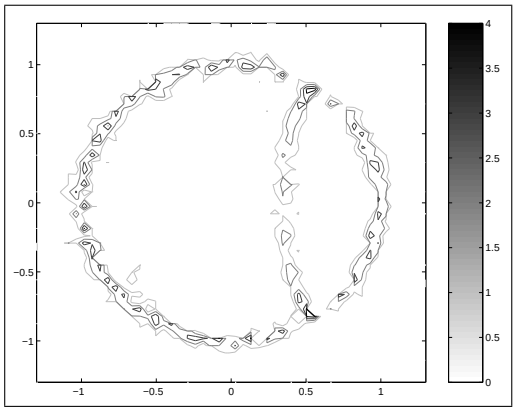

Figure 1. *The* log *empirical pooled distribution*

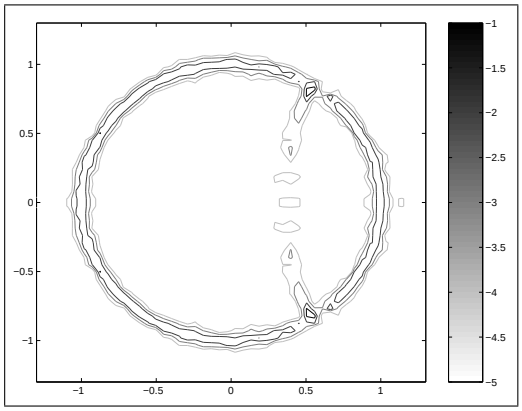

Figure 2. *The* log *condensed density estimated by MonteCarlo integration on* 100 *samples*

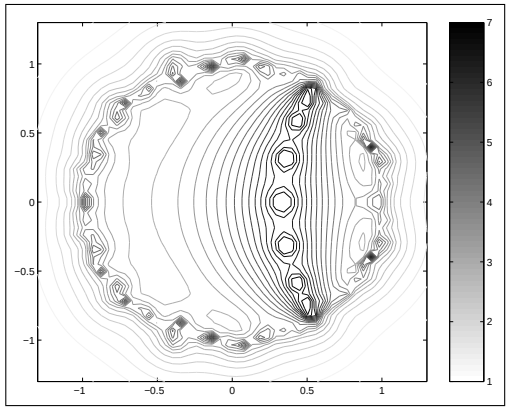

Figure 3. *The* log *pseudospectrum*

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