EXPANSION-CONTRACTION MATRICES AND CONTRACTIBILITY MATRICES IN THE INCLUSION PRINCIPLE FOR LINEAR TIME-INVARIANT SYSTEMS

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Abstract. A central problem for the application of the inclusion principle in the context of analysis and control of large-scale dynamic systems is the selection of the expansion-contraction matrices and contractibility matrices. All previous results enable such selection only partially because of the usage of the forms of the expansion-contraction matrices and contractibility matrices corresponding only with some particular cases. In this paper we parameterize all expansion-contraction matrices and contractibility matrices explicitly. Our results provide the full freedom in the selection of the expansion-contraction matrices and contractibility matrices in system analysis and design under inclusion principle.

1. Introduction. The *Inclusion Principle*, developed in 1980's in the context of analysis and control of large-scale dynamic systems [9, 10, 11, 12], establishes a mathematical framework in which two dynamic systems with different dimensions may have related behavior. Under the inclusion principle, a "big" system can be built from a "small" system via an *expansion* process, in such a way that the "big" system contains the essential information about the behavior of the "small" system, furthermore, this information can be extracted from the "big" system by a contraction process. The inclusion principle has been studied extensively and found many applications in the design of decentralized controllers [6, 7, 8, 13, 1, 14, 15].

Consider a pair of linear time-invariant systems

$$\dot{x} = Ax + Bu, \quad y = Cx, \tag{1.1}$$

and

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}\tilde{u}, \quad \tilde{y} = \tilde{C}\tilde{x}, \tag{1.2}$$

where $x(t) \in \mathbf{R}^n$, $u(t) \in \mathbf{R}^m$, $y(t) \in \mathbf{R}^l$ are the state, input and output of system (1.1) at time $t \geq 0$, and $\tilde{x}(t) \in \mathbf{R}^{\tilde{n}}$, $\tilde{y}(t) \in \mathbf{R}^{\tilde{m}}$, $\tilde{y}(t) \in \mathbf{R}^{\tilde{l}}$ are those of system (1.2), $A \in \mathbf{R}^{n\times n}$, $B \in \mathbf{R}^{n\times m}$, $C \in \mathbf{R}^{l\times n}$ and $\tilde{A} \in \mathbf{R}^{\tilde{n}\times\tilde{n}}$, $\tilde{B} \in \mathbf{R}^{\tilde{n}\times\tilde{m}}$, $\tilde{C} \in \mathbf{R}^{\tilde{l}\times\tilde{n}}$ are constant. Suppose

$$n \leq \tilde{n}, \quad m \leq \tilde{m}, \quad l \leq \tilde{l},$$

i.e., system (1.1) is "smaller" than system (1.2). Denote $x(t; x_0, u)$ and $y[x(t)]$ the state behavior and the corresponding output of system (1.1) for a fixed input $u(t)$ and for an initial state $x(0) = x_0$, respectively. Similar notations $\tilde{x}(t; \tilde{x}_0, \tilde{u})$ and $\tilde{y}[\tilde{x}(t)]$ are used for the state behavior and output of system (1.2).

Let's link systems (1.1) and (1.2) through the following transformations:

$$V : \mathbf{R}^n \longrightarrow \mathbf{R}^{\tilde{n}}, \quad R : \mathbf{R}^m \longrightarrow \mathbf{R}^{\tilde{m}}, \quad T : \mathbf{R}^l \longrightarrow \mathbf{R}^{\tilde{l}}, \tag{1.3}$$

where

$$\mathrm{rank}(V) = n, \quad \mathrm{rank}(R) = m, \quad \mathrm{rank}(T) = l. \tag{1.4}$$

Denote the unique pseudoinverses of V, R and T by $V^{(+)}$, $R^{(+)}$ and $T^{(+)}$, respectively.

DEFINITION 1.1. *(The Inclusion Principle) The system (1.2) includes the system (1.1), that is, system (1.1) is included by system (1.2), if there exists a triplet (V, R, T) satisfying (1.3) and (1.4) such that, for any initial state x_0 and any fixed $u(t)$ of system (1.1), the choice*

$$\tilde{x}_0 = Vx_0, \ \tilde{u}(t) = Ru(t), \ \forall t \geq 0 \tag{1.5}$$

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of the initial state $\tilde{x}_0$ and input $\tilde{u}(t)$ of the system (1.2) implies

$$x(t;x_0,u) = V^{(+)}\tilde{x}(t;\tilde{x}_0,\tilde{u}),\ y[x(t)] = T^{(+)}\tilde{y}[\tilde{x}(t)],\ \forall t \geq 0. \tag{1.6}$$

If the system (1.2) includes the system (1.1), then system (1.2) is said to be an expansion of the system (1.1) and system (1.1) is a contraction of system (1.2).

DEFINITION 1.2. *The control law $\tilde{u} = -\tilde{K}\tilde{x}$ for system (1.2) is contractible to the control law $u = -Kx$ for system (1.1) if the choice*

$$\tilde{x}_0 = Vx_0,\ \tilde{u}(t) = Ru(t)$$

implies

$$Kx(t;x_0,u) = R^{(+)}\tilde{K}\tilde{x}(t;\tilde{x}_0,\tilde{u}) \tag{1.7}$$

for all $t \geq 0$, any initial state x_0 and any fixed input $u(t)$ of system (1.1).

For the expansion-contraction and contracbility between systems (1.1) and (1.2), the conditions are provided in the following theorem.

THEOREM 1.3. *[11] Given systems (1.1) and (1.2) and transformations V, R and T satisfying (1.3) and (1.4).*

(i) The system (1.2) is an expansion of the system (1.1) if and only if for all $i = 1, 2, \cdots, \tilde{n}$

$$\begin{cases} V^{(+)}(\tilde{A} - VAV^{(+)})^{i}V = 0, \\ V^{(+)}(\tilde{A} - VAV^{(+)})^{i-1}(\tilde{B}R - VB) = 0, \\ (T^{(+)}\tilde{C} - CV^{(+)})(\tilde{A} - VAV^{(+)})^{i-1}V = 0, \\ (T^{(+)}\tilde{C} - CV^{(+)})(\tilde{A} - VAV^{(+)})^{i-1}(\tilde{B}R - VB) = 0. \end{cases} \tag{1.8}$$

(ii) A control law $\tilde{u} = -\tilde{K}\tilde{x}$ for system (1.2) is contractible to the control law $u = -Kx$ for system (1.1) if and only if for all $i = 1, 2, \cdots, \tilde{n}$

$$\begin{cases} (R^{(+)}\tilde{K} - KV^{(+)})(\tilde{A} - VAV^{(+)})^{i-1}V = 0, \\ (R^{(+)}\tilde{K} - KV^{(+)})(\tilde{A} - VAV^{(+)})^{i-1}(\tilde{B}R - VB) = 0. \end{cases} \tag{1.9}$$

The applications of inclusion principle rely on the choice of system (1.2) which has the freedom of the selection of the matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{K}$. It is easy to see that the conditions given in Theorem 1.3 are difficult to verify in practice. Therefore, as pointed out in [13, 1, 14, 15], there is not a systematic procedure for choosing the matrices of system (1.2). This leads that only a few simple standard choices have been comonly used in existing research, while the exploitation of the degree of freedom offered by the selection of system (1.2) has been considered as an interesting study issue [10]. In this direction, by introducing appropriate changes of the basis in the expansion-contraction process and then taking

$$V = \begin{bmatrix} I_{n_1} & 0 & 0 \\ 0 & I_{n_2} & 0 \\ 0 & I_{n_2} & 0 \\ 0 & 0 & I_{n_3} \end{bmatrix},\ R = \begin{bmatrix} I_{m_1} & 0 & 0 \\ 0 & I_{m_2} & 0 \\ 0 & I_{m_2} & 0 \\ 0 & 0 & I_{m_3} \end{bmatrix},\ T = \begin{bmatrix} I_{l_1} & 0 & 0 \\ 0 & I_{l_2} & 0 \\ 0 & I_{l_2} & 0 \\ 0 & 0 & I_{l_3} \end{bmatrix} \tag{1.10}$$

with

$$n_1 + n_2 + n_3 = n,\ m_1 + m_2 + m_3 = m,\ l_1 + l_2 + l_3 = l,$$

$$n_1 + 2n_2 + n_3 = \tilde{n},\ m_1 + 2m_2 + m_3 = \tilde{m},\ l_1 + 2l_2 + l_3 = \tilde{l},$$

two new kinds of expansion-contraction matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and contractibility matrix $\tilde{K}$ in system (1.2) has been recently presented in [1]. The results in [1] stand for the art-of-state of the study on the selection of expansion-contraction matrices and contractiblity matrices under inclusion principle. However, they are incomplete in the following sense:

- The necessary and sufficient conditions given in [1] (i.e., its Theorem 3.9) involve matrix powers and different block matrices are interwound, thus, it can not lead to the explicit parameterization of all expansion-contraction matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and contractiblity matrix $\tilde{K}$ in system (1.2). In order to exploit the full freedom under inclusion principle, it is necessary to parameterize all expansion-contraction matrices and contractiblity matrices explicitly .

Motivated by the above observation, in this paper we will characterize the expansion-contraction matrices and contractiblity matrices with general setting of transformations V, R, T under inclusion principle and parameterize them explicitly.

2. Explicit Parameterization of Expansion-Contraction matrices and Contractiblity Matrices. The purpose of this section is to parameterize all expansion-contraction matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and contractibility matrix $\tilde{K}$ in system (1.2) under inclusion principle.

THEOREM 2.1. *Given system (1.1) and transformations V, R, T satisfying (1.3) and (1.4). Let the QR factorizations of V, R and T are given by*

$$
\left\{
\begin{array}{ll}
\begin{bmatrix} \mathcal{U} & U \end{bmatrix}^T V = \begin{array}{c} \begin{bmatrix} V_{11} \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \}n \\ \}\tilde{n}-n \end{array}, & \mathcal{U} \in \mathbf{R}^{\tilde{n}\times n}, \quad U \in \mathbf{R}^{\tilde{n}\times(\tilde{n}-n)}, \\[2ex]
\begin{bmatrix} \mathcal{P} & P \end{bmatrix}^T R = \begin{array}{c} \begin{bmatrix} R_{11} \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \}m \\ \}\tilde{m}-m \end{array}, & \mathcal{P} \in \mathbf{R}^{\tilde{m}\times m}, \quad P \in \mathbf{R}^{\tilde{m}\times(\tilde{m}-m)}, \\[2ex]
\begin{bmatrix} \mathcal{Q} & Q \end{bmatrix}^T T = \begin{array}{c} \begin{bmatrix} T_{11} \\ 0 \end{bmatrix} \end{array} \begin{array}{l} \}l \\ \}\tilde{l}-l \end{array}, & \mathcal{Q} \in \mathbf{R}^{\tilde{l}\times l}, \quad Q \in \mathbf{R}^{\tilde{l}\times(\tilde{l}-l)},
\end{array}
\right. \tag{2.1}
$$

where $\begin{bmatrix} \mathcal{U} & U \end{bmatrix}$, $\begin{bmatrix} \mathcal{P} & P \end{bmatrix}$ and $\begin{bmatrix} \mathcal{Q} & Q \end{bmatrix}$ are orthogonal, and V_{11}, R_{11} and T_{11} are nonsingular. Then the system (1.2) is an expansion of the system (1.1) if and only if

$$
\left\{
\begin{array}{l}
\tilde{A} = \begin{bmatrix} V & UW \end{bmatrix} \begin{bmatrix} A & 0 & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & 0 & \tilde{A}_{33} \end{bmatrix} \begin{bmatrix} V^{(+)} \\ (UW)^T \end{bmatrix}, \\
\tilde{B} = \begin{bmatrix} V & UW \end{bmatrix} \begin{bmatrix} B & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \\ 0 & \tilde{B}_{32} \end{bmatrix} \begin{bmatrix} R^{(+)} \\ P^T \end{bmatrix}, \\
\tilde{C} = \begin{bmatrix} T & Q \end{bmatrix} \begin{bmatrix} C & 0 & \tilde{C}_{13} \\ \tilde{C}_{21} & \tilde{C}_{22} & \tilde{C}_{23} \end{bmatrix} \begin{bmatrix} V^{(+)} \\ (UW)^T \end{bmatrix},
\end{array}
\right. \tag{2.2}
$$

where $W \in \mathbf{R}^{(\tilde{n}-n)\times(\tilde{n}-n)}$ is an arbitrary orthogonal matrix, μ is an arbitrary integer between 0 and $\tilde{n}-n$, and $\tilde{A}_{13} \in \mathbf{R}^{n\times(\tilde{n}-n-\mu)}$, $\tilde{A}_{21} \in \mathbf{R}^{\mu\times n}$, $\tilde{A}_{22} \in \mathbf{R}^{\mu\times\mu}$, $\tilde{A}_{23} \in \mathbf{R}^{\mu\times(\tilde{n}-n-\mu)}$, $\tilde{A}_{33} \in \mathbf{R}^{(\tilde{n}-n-\mu)\times(\tilde{n}-n-\mu)}$, $\tilde{B}_{12} \in \mathbf{R}^{n\times(\tilde{m}-m)}$, $\tilde{B}_{21} \in \mathbf{R}^{\mu\times m}$, $\tilde{B}_{22} \in \mathbf{R}^{\mu\times(\tilde{m}-m)}$, $\tilde{B}_{32} \in \mathbf{R}^{(\tilde{n}-n-\mu)\times(\tilde{m}-m)}$, $\tilde{C}_{13} \in \mathbf{R}^{l\times(\tilde{n}-n-\mu)}$, $\tilde{C}_{21} \in \mathbf{R}^{(\tilde{l}-l)\times n}$, $\tilde{C}_{22} \in \mathbf{R}^{(\tilde{l}-l)\times\mu}$ and $\tilde{C}_{23} \in \mathbf{R}^{(\tilde{l}-l)\times(\tilde{n}-n-\mu)}$ are arbitrary matrices.

THEOREM 2.2. *Given systems (1.1) and (1.2) and transformations V, R, T satisfying (1.3) and (1.4). Let $\begin{bmatrix} \mathcal{U} & U \end{bmatrix}$, $\begin{bmatrix} \mathcal{P} & P \end{bmatrix}$, $\begin{bmatrix} \mathcal{Q} & Q \end{bmatrix}$, V_{11} and R_{11} are the same as those in Theorem 2.1 and (2.1). Define*

$\left[\ \mathcal{U}\quad U\ \right]^T \tilde{A} \left[\ \mathcal{U}\quad U\ \right]$ *and* $\left[\ \mathcal{U}\quad U\ \right]^T \tilde{B} \left[\ \mathcal{P}\quad P\ \right]$ *by*

$$
(2.3)\qquad \begin{cases}
\left[\ \mathcal{U}\quad U\ \right]^T \tilde{A} \left[\ \mathcal{U}\quad U\ \right] = \begin{array}{cc} n & \tilde{n}-n \end{array} \\
\qquad\qquad\qquad \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix} \begin{array}{l} \}n \\ \}\tilde{n}-n \end{array}, \\
\left[\ \mathcal{U}\quad U\ \right]^T \tilde{B} \left[\ \mathcal{P}\quad P\ \right] = \begin{array}{cc} m & \tilde{m}-m \end{array} \\
\qquad\qquad\qquad \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{bmatrix} \begin{array}{l} \}n \\ \}\tilde{n}-n \end{array}, \\
\left[\ \mathcal{Q}\quad Q\ \right]^T \tilde{C} \left[\ \mathcal{U}\quad U\ \right] = \begin{array}{cc} n & \tilde{n}-n \end{array} \\
\qquad\qquad\qquad \begin{bmatrix} \hat{C}_{11} & \hat{C}_{12} \\ \hat{C}_{21} & \hat{C}_{22} \end{bmatrix} \begin{array}{l} \}l \\ \}\tilde{l}-l \end{array}.
\end{cases}
$$

Let orthogonal matrix $W \in \mathbf{R}^{(\tilde{n}-n)\times(\tilde{n}-n)}$ *and the integer* μ *be such that* $(\hat{A}_{22}, \left[\ \hat{A}_{21}V_{11}\quad \hat{B}_{21}R_{11}\ \right])$ *is in its controllable stair-case form as follows*

$$
(2.4)\qquad \begin{cases}
W^T \hat{A}_{22} W = \begin{array}{cc} \mu & \tilde{n}-n-\mu \end{array} \\
\qquad\qquad \begin{bmatrix} \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & \tilde{A}_{33} \end{bmatrix} \begin{array}{l} \}\mu \\ \}\tilde{n}-n-\mu \end{array}, \\
W^T \left[\ \hat{A}_{21}V_{11} \mid \hat{B}_{21}R_{11}\ \right] = \begin{array}{cc} n & m \end{array} \\
\qquad\qquad \begin{bmatrix} \tilde{A}_{21} & \tilde{B}_{21} \\ 0 & 0 \end{bmatrix} \begin{array}{l} \}\mu \\ \}\tilde{n}-n-\mu \end{array}, \\
(\tilde{A}_{22}, \left[\ \tilde{A}_{21}\quad \tilde{B}_{21}\ \right]) \text{ is controllable.}
\end{cases}
$$

Then the control law $\tilde{u} = -\tilde{K}\tilde{x}$ *for system (1.2) is contractible to the control law* $u = -Kx$ *for system (1.1) if and only if*

$$
\tilde{K} = \left[\ R\quad P\ \right] \begin{bmatrix} K & 0 & \tilde{K}_{13} \\ \tilde{K}_{21} & \tilde{K}_{22} & \tilde{K}_{23} \end{bmatrix} \begin{bmatrix} V^{(+)} \\ (UW)^T \end{bmatrix},
$$

where $\tilde{K}_{13} \in \mathbf{R}^{m\times(\tilde{n}-n-\mu)}$, $\tilde{K}_{21} \in \mathbf{R}^{(\tilde{m}-m)\times n}$, $\tilde{K}_{22} \in \mathbf{R}^{(\tilde{m}-m)\times \mu}$ *and* $\tilde{K}_{23} \in \mathbf{R}^{(\tilde{m}-m)\times(\tilde{n}-n-\mu)}$ *are arbitrary.*

It has been addressed in [1] that our ability to use generalized (system) decompositions depends crucially not only on the choice of the transformation matrices V, R and T, but also on the selection of the expansion-contraction matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and the contractibility matrix $\tilde{K}$ in system (1.2). In this context, the importance of the selection of matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ and $\tilde{K}$ in system (1.2) is clear. All previous results enable such selection only partially because of the usage of the forms of matrices $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ in system (1.2) corresponding only with some particular cases. Theorems 2.1 and 2.2 have explicitly parameterized all admissible expansion-contraction matrices $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ and contractibility matrix $\tilde{K}$ in system (1.2) and thus provide full freedom under inclusion principle. Therefore, the significance of Theorems 2.1 and 2.2 is obvious.

3. Conclusion. In this paper we have given the explicit parameterization of the expansion-contraction matrices and contractibility matrices in the inclusion principle in the context of analysis and control of large-scale dynamic systems, which leads that the full freedom in the selection of the expansion-contraction matrices and contractibility matrices can be fully exploited in system analysis and design.

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