Graph cut, convex relaxation and continuous max-flow problem

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Context of this presentation

- Overview over recent combinatorial graph cut methods and convex relaxation methods in imaging science with focus on interface problems
- Category 1: Problems that can be solved exactly
  - Always direct relation between graph cut and convex relaxations via continuous max-flow
- Category 2: Problems that can only be solved approximately (NP-hard)
  - Very good approximations can be obtained via different convex relaxations
  - Dual problems can be interpreted as continuous max-flow problems
- Efficient convex max-flow algorithms can be derived for all problems

Interface problems

Interface problems exists everywhere in science and technology. For imaging and vision, it is somehow classical:

- Mumford-Shal model (Mumford-Shah-1989)
- GAC model (Caselles-Kimmel-Sapiro-1997)
- Chan-Vese model (Chan-Vese-2001)

How to solve these interface problems?

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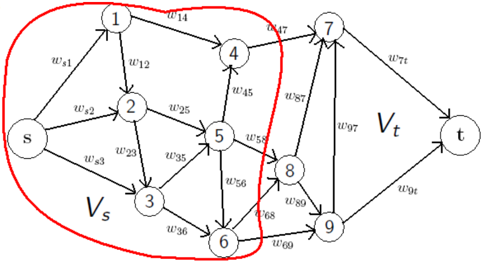
- Chan-Vese model (Chan-Vese-2001)

How to solve these interface problems?

- active contour (Kass-Witkin-Terzopoulos-1998)
- level set (Osher-Sethian-1988)
- phase-field (Modica-Mortola-1977, Ambresio-Tortorelli-1990)
- ...

Introduction to Max-Flow / Min-Cut



Ref: Ford and D. R. Fulkerson, 1962.

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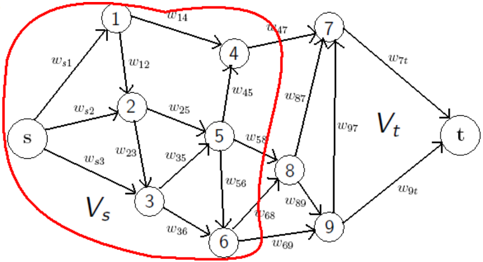

(V_s, V_t) is a cut, w_{ij} = cost of cutting edge(i, j)
cost of cut $c(V_s, V_t) = \sum_{i \in V_s, j \in V_t} w_{ij}$

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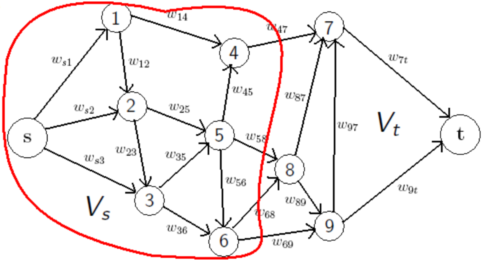

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Min-cut: find cut of minimum cost,

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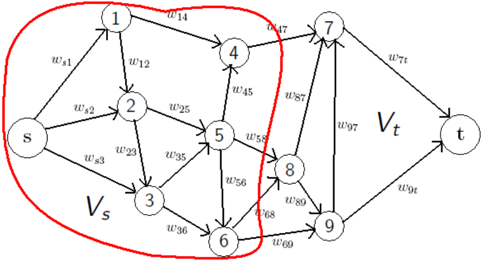

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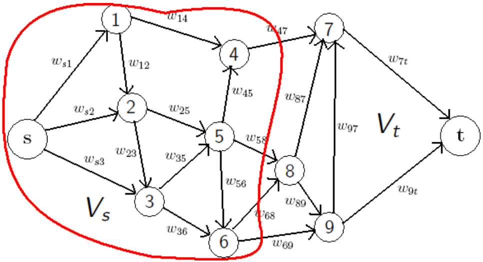

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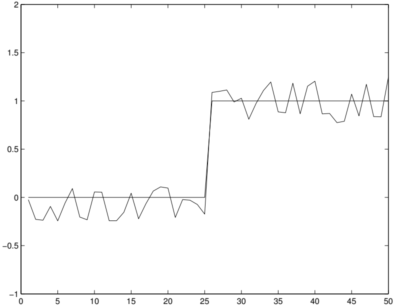
Min-cut: find cut of minimum cost,
Max-Flow: Find the maximum amount of flow from s to t.

Max-flow = min-cut.

Ref: Ford and D. R. Fulkerson, 1962.

Graph-cut for image segmentation

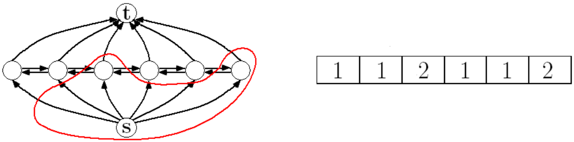
A simple 1d signal $I(x)$:



Graph-cut for images: Boykov-Kolmogorov (2001).

Graph-cut for image segmentation

The graph, a graph-cut and its corresponding label:

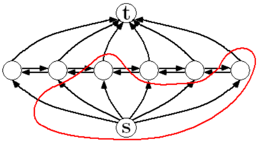


Popular "capacity" choices: (Chan-Vese-2001)

$$w_{s,p} = |I(p) - c_1|^2, w_{t,p} = |I(p) - c_2|^2, c_1 = 0, c_2 = 1, \quad w(p,q) = \alpha.$$

Graph-cut for image segmentation

The graph, a graph-cut and its corresponding label:



1	1	2	1	1	2

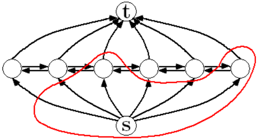
Popular "capacity" choices: (Chan-Vese-2001)

$$w_{s,p} = |I(p) - c_1|^2, w_{t,p} = |I(p) - c_2|^2, c_1 = 0, c_2 = 1, \;\; w(p,q) = \alpha.$$

More generally

$$w_{s,p} = f_1(p), \;\; w_{t,p} = f_2(p), \;\; w(p,q) = \alpha \text{ or } g(p,q) \text{ (edge force)}.$$

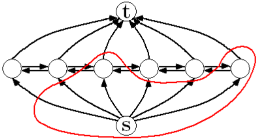
Relation with k-mean ($\alpha = 0$ and unknown c_i)



1	1	2	1	1	2

- Given c_1 and c_2.

Relation with k-mean ($\alpha = 0$ and unknown c_i)



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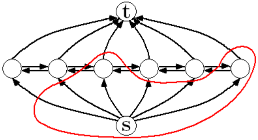
- Given c_1 and c_2.
- use cut (threshold) to get Ω_1 and Ω_2.

Relation with k-mean ($\alpha = 0$ and unknown c_i)

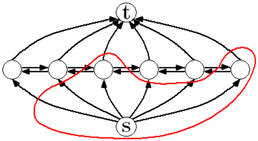


1	1	2	1	1	2

- Given c_1 and c_2.
- use cut (threshold) to get Ω_1 and Ω_2.
- update

$$c_i = \frac{\int_{\Omega_i} I(x)}{Area(\Omega_i)}, i = 1, 2.$$

Relation with k-mean ($\alpha = 0$ and unknown c_i)



1	1	2	1	1	2

- Given c_1 and c_2.
- use cut (threshold) to get Ω_1 and Ω_2.
- update

$$c_i = \frac{\int_{\Omega_i} I(x)}{Area(\Omega_i)}, i = 1, 2.$$

- go to the next iteration.

k-mean is a non-regularized Chan-Vese model.

k-mean model ($\alpha = 0$ and unknown c_i)

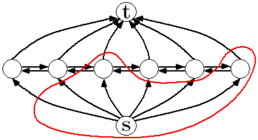
k-mean algorithm is an alternating minimization procedure for:

$$\min_{c_i,\Omega_i} \sum_{i=1}^{n} \int_{\Omega_i} (I(x) - c_i)^2.$$

This formulation is in the continuous setting.

Ref: K-means clustering, http://en.wikipedia.org/wiki/K-means_clustering.

Regularized Graph-cut: $\alpha \neq 0$



1	1	2	1	1	2

The "virtual graph and the corresponding label function $u(p), p = 1, 2, \cdots$.

Costs:

$$w_{s,p} = |I(p) - c_1|^2, w_{t,p} = |I(p) - c_2|^2, w_{p,q} = \alpha.$$

The corresponding minimization problem is: ($N(p)$ – neighbors of p)

$$\min_{u(p)\in\{1,2\}} \sum_{p\in\Omega_1} |I(p)-c_1|^2 + \sum_{p\in\Omega_2} |I(p)-c_2|^2 + \alpha \sum_{p} \sum_{q\in N(p)} |u(p)-u(q)|.$$

Discrete vs continuous

Discrete minimization:

$$\min_{u(p)\in\{0,1\}} \sum_{p\in\Omega_1} |I(p)-c_1|^2 + \sum_{p\in\Omega_2} |I(p)-c_2|^2 + \alpha \sum_{p} \sum_{q\in N(p)} |u(p)-u(q)|.$$

Continuous minimization:

$$\min_{u(x)\in\{0,1\}} \int_{\Omega_1} |I(x)-c_1|^2 + \int_{\Omega_2} |I(x)-c_2|^2 + 4\alpha \int_{\Omega} |Du|.$$

$$\min_{u(x)\in\{0,1\}} \int_{\Omega} |I(x)-c_1|^2 (1-u) + \int_{\Omega} |I(x)-c_2|^2 u + 4\alpha \int_{\Omega} |Du|.$$

Higher dimensional problems

A graph for 2D images:

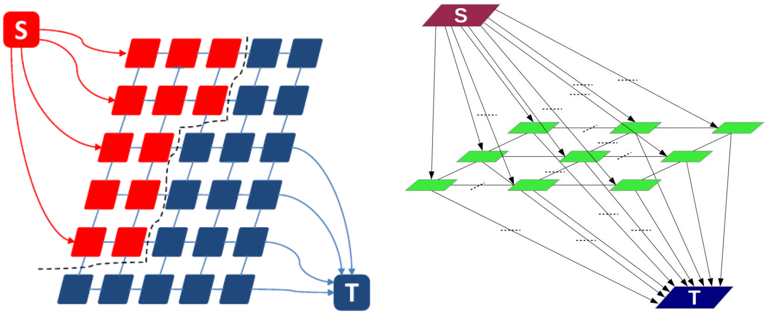


Figure : Graph used for discrete 2D binary labeling

Two-phase Min-cut – Discretized setting

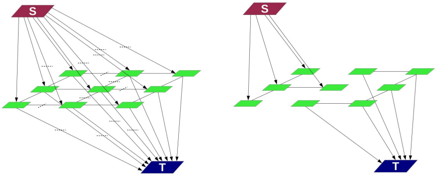


Figure : Graph and cut for discrete binary labeling

It is easy to see the cost of a cut ($u(p) = 0$ or 1). A minimum cut is to find u for:

$$\min_{u \in \{0,1\}} \sum_{p \in \mathcal{P}} f_1(p)(1 - u(p)) + f_2(p)u(p) + \sum_{p \in \mathcal{P}} \sum_{q \in \mathcal{N}_p^k} g(p,q)|u(p) - u(q)|.$$

Capacity:

$$w_{s,p} = f_1(p), \; w_{t,p} = f_2(p), \; w_{p,q} = g(p,q).$$

Ref: $\mathcal{N}_p^k$ is the k-neighborhood of $p \in \mathcal{P}$.

Two-phase Min-cut – corresponding continuous setting

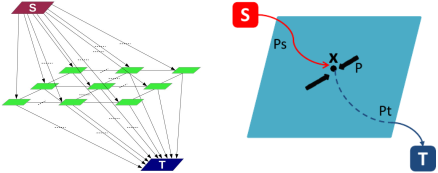


Figure : Graph used for discrete and continuous binary labeling

A "continuous" minimum cut is to solve:

$$\min_{u \in \{0,1\}} \int_\Omega f_1(x)(1-u(x)) + f_2(x)u(x) + g_1(x)|D_1 u(x)| + g_2(x)|D_2 u(x)|.$$

Capacity:

$$w_s(x) = f_1(x), \; w_t(x) = f_2(x), \; w_1(x) = g_1(x), \quad w_2(x) = g_2(x).$$

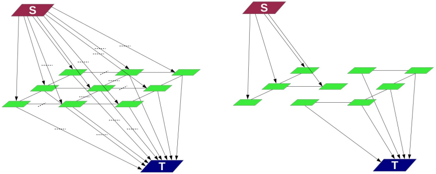


Figure : Graph used for discrete binary labeling

Max-flow formulation

$$\max_{p_s, p_t, q} \sum_{v \in \mathcal{V} \setminus \{s,t\}} p_s(v)$$

subject to

$$\begin{aligned}
|q(v,u)| &\leq g(v,u), \quad && \forall (v,u) \in \mathcal{V} \times \mathcal{V} \\
0 \leq p_s(v) &\leq f_1(v), \quad && \forall v \in \mathcal{V} \setminus \{s,t\}; \\
0 \leq p_t(v) &\leq f_2(v), \quad && \forall v \in \mathcal{V} \setminus \{s,t\}; \\
\Big(\sum_{u \in N(v)} q(v,u) \Big) - p_s(v) + p_t(v) &= 0, \quad && \forall v \in \mathcal{V} \setminus \{s,t\}; .
\end{aligned}$$

Continuous Max-Flow

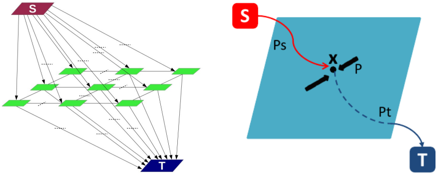


Figure : Discrete (left) vs. Continuous (right)

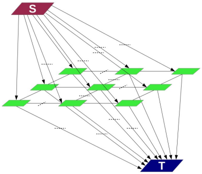
Continuous max-flow formulation

$$\sup_{p_s, p_t, q} \int_\Omega p_s(x)\, dx$$

subject to

$$\begin{aligned} |q_1(x)| \leq g_1(x);\ \ q_2(x)| &\leq g_2(x), && \forall x \in \Omega; \\ 0 \leq p_s(x) &\leq f_1(x), && \forall x \in \Omega; \\ 0 \leq p_t(x) &\leq f_2(x), && \forall x \in \Omega; \\ \operatorname{div} q(x) - p_s(x) + p_t(x) &= 0, && \text{a.e. } x \in \Omega. \end{aligned}$$

Related: (G. Strang (1983)).



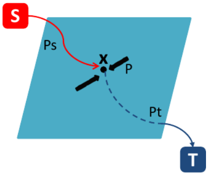


Figure : Discrete (left) vs. Continuous (right)

Continuous max-flow formulation

$$\sup_{p_s, p_t, q} \int_\Omega p_s(x)\, dx$$

subject to

$$\begin{aligned} |q(x)| = \sqrt{q_1^2(x) + q_2^2(x)} \; &\leq \; g(x), \quad && \forall x \in \Omega; \\ 0 \leq p_s(x) \; &\leq \; f_1(x), \quad && \forall x \in \Omega; \\ 0 \leq p_t(x) \; &\leq \; f_2(x), \quad && \forall x \in \Omega; \\ \operatorname{div} q(x) - p_s(x) + p_t(x) \; &= \; 0, \quad && \text{a.e. } x \in \Omega. \end{aligned}$$

Connection: Continuous Max-Flow and Min-Cut

Lagrange multiplier u for flow conservation condition

$$\operatorname{div} q(x) - p_s(x) + p_t(x) \;=\; 0, \qquad \text{a.e. } x \in \Omega.$$

yields primal-dual formulation

$$\sup_{p_s, p_t, q} \; \inf_{u} \; \int_\Omega p_s + u\big(\operatorname{div} q - p_s + p_t\big)\, dx$$

$$\text{s.t.} \quad p_s(x) \;\leq\; f_1(x)\,, \quad p_t(x) \leq f_2(x)\,, \quad |q(x)| \;\leq\; g(x)\,.$$

Optimizing for flows p_s, p_t, q results in:

$$\min_{u \in [0,1]} \; \int_\Omega f_1(x)(1 - u(x)) + f_2(x) u(x)\, dx + g(x) \left|\nabla u(x)\right| \, dx\,.$$

This is exactly the same model as the model in CEN (2006). [1]

[1] T. F. Chan and S. Esedoglu and M. Nikolova: Algorithms for finding global minimizers of image segmentation and denoising models, SIAM J. Appl. Math., 66, 1632–1648,(2006)

Three problems

$$\boxed{\min_{u(x)\in\{0,1\}} \int_\Omega f_1(1-u) + f_2 u + g(x)|\nabla u| dx.}$$

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$$\max_{p_s,p_t,q} \int_\Omega p_s dx \text{ subject to:}$$

$$p_s(x) \le f_1(x),\ \ p_t(x) \le f_2(x),\ \ |p(x)| \le g(x),$$

$$\mathrm{div} p(x) - p_s(x) + p_t(x) = 0.$$

Three problems

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Three problems

PCLMS or Binary LM (Lie-Lysaker-T.,2005):

$$\min_{u(x)\in\{0,1\}} \int_\Omega f_1(1-u) + f_2 u + g(x)|\nabla u| dx.$$

Convex problem (CEN, (Chan-Esdoglu-Nikolova,2006))

$$\min_{u(x)\in[0,1]} \int_\Omega f_1(1-u) + f_2 u + g(x)|\nabla u| dx.$$

Graph-cut (Boykov-Kolmogorov,2001)

$$\max_{p_s,p_t,q} \int_\Omega p_s dx \text{ subject to:}$$

$$p_s(x) \le f_1(x),\ \ p_t(x) \le f_2(x),\ \ |p(x)| \le g(x),$$

$$\mathrm{div} p(x) - p_s(x) + p_t(x) = 0.$$

Remarks

The following approaches are solving the same problem, but did not know each other:

- max-flow and min-cut.
- Chan-Esedougla-Nikolova 2006 (convex relaxation approach)
- Binary Level set methods and PCLSM (piecewise constant level set method)
- A cut is nothing else, but the Lagrangian multiplier for the flow conservation constraint!!!

Continuous Max-Flow: Remarks

- Min-cut problem is minimizing an energy functional. (Many existing algorithms) Are not using the decent (gradient) info of the energy.
- Continuous max-flow/min-cut is a convex minimization problem. A lot of choices, can use decent (gradient) info.

Continuous Max-Flow: How to solve it (Only 2-phase case)?

- Popular (discrete) Min-cut algorithms: Augmented Path. Push-relabel, etc,
- Available continuouse max-flow/Min-cut approaches: Split-Bregman, Augmented Lagrangian, Primal-Dual approaches. We can use these approach to solve the convex min-cut problem.

Multiplier-Based Maximal-Flow Algorithm

Augmented lagrangian functional (Glowinski & Le Tallec, 1989)

$$L_c(p_s, p_t, q, \lambda) := \int_\Omega p_s\, dx + \lambda\big(\operatorname{div} q - p_s + p_t\big) - \frac{c}{2}|\operatorname{div} q - p_s + p_t|^2\, dx.$$

minmax subject to:

$p_s(x) \leq f_1(x)\,, \quad p_t(x) \leq f_2(x)\,, \quad |q(x)| \leq g(x)$

ADMM algorithm: For k=1,... until convergence, solve

$$q^{k+1} := \arg \max_{\|q\|_\infty \leq \alpha} L_c(p_s^k, p_t^k, q, \lambda^k)$$

$$p_s^{k+1} := \arg \max_{p_s(x) \leq f_1(x)} L_c(p_s, p_t^k, q^{k+1}, \lambda^k)$$

$$p_t^{k+1} := \arg \max_{p_t(x) \leq f_2(x)} L_c(p_s^{k+1}, p_t, q^{k+1}, \lambda^k)$$

$$\lambda^{k+1} = \lambda^k - c\,(\operatorname{div} q^{k+1} - p_s^{k+1} + p_t^{k+1})$$

Other algorithms for solving the relaxed problem: add $p = \nabla u$

- Bresson et. al.
  - fix μ^k and solve ROF problem

$$\lambda^{k+1} := \arg\min_{\lambda} \left\{ \alpha \int_{\Omega} |\nabla \lambda(x)| \, dx + \frac{1}{2\theta} \|\lambda(x) - \mu^k(x)\|^2 \right\}$$

  - fix λ^{k+1} and solve

$$\mu^{k+1} := \arg \min_{\mu \in [0,1]} \left\{ \frac{1}{2\theta} \|\mu(x) - \lambda^{k+1}\|^2 + \int_{\Omega} \mu(x) \big(f_1(x) - f_2(x)\big) \, dx \right\}$$

- Goldstein-Osher: Split Bregman / augmented lagrangian

Convergence

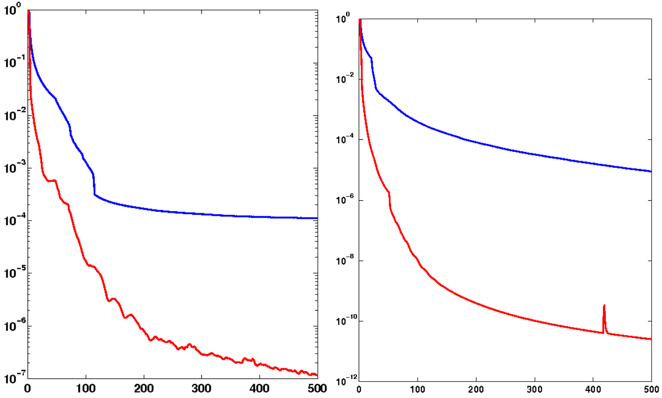
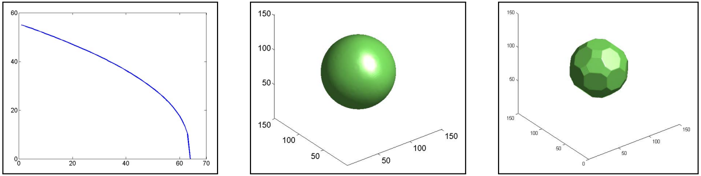


Figure : Red line: max-flow algorithm. Blue line: Splitting algorithm (Bresson et. al. 2007)

Metrication error, Parallel, GPU, ...



Experiment of mean-curvature driven 3D surface evolution (volume size: 150X150X150 voxels). (a) The radius plot of the 3D ball evolution driven by its mean-curvature flow, which is computed by the proposed continuous max-flow algorithm; its function is theoretically $r(t) = \sqrt{C - 2t}$. (b) The computed 3D ball at one discrete time frame, which fits a perfect 3D ball shape. This is in contrast to (c), the computation result by graph cut [15] with a 3D 26-connected graph. The computation time of the continuous max-flow algorithm for each discrete time evolution is around 1 sec., which is faster than the graph cut method (120 sec.)

Ref: Y. Yuan, E. Ukwatta, X. Tai, A. Fenster, and C. Schnorr. A fast global optimization-based approach to evolving contours with generic shape prior. Technical report, also UCLA Tech. Report CAM 12-38, 2012.

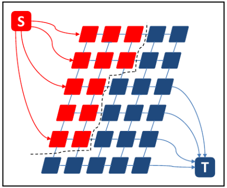
Metrication error, Parallel, GPU, ...

- Fully parallel, easy GPU implementation.
- linear grow of computational cost (per iteration): 2D, 3D, ...

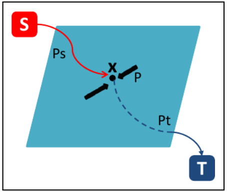
Gamma convergence

When $h \mapsto 0$, the energy at the minimizer on the discrete graph converges to energy at the minimizer of the continuous problem both for isotropic and anisotropic TV:

$$TV(u) = \int_\Omega (|u_x| + |u_y|)dx, \qquad TV(u) = \int_\Omega \sqrt{|u_x|^2 + |u_y|^2}dx.$$



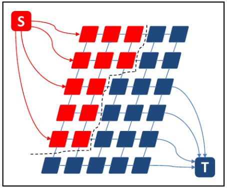
$$\xrightarrow{\Gamma}$$



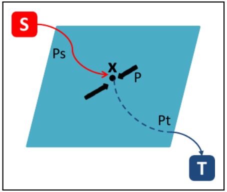
Gamma convergence

When $h \mapsto 0$, the energy at the minimizer on the discrete graph converges to energy at the minimizer of the continuous problem both for isotropic and anisotropic TV:

$$TV(u) = \int_\Omega (|u_x| + |u_y|)dx, \qquad TV(u) = \int_\Omega \sqrt{|u_x|^2 + |u_y|^2}dx.$$



$$\xrightarrow{\Gamma}$$



- Anisotropic TV:

Gamma convergence

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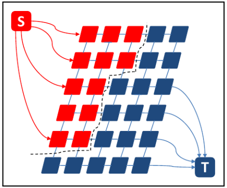
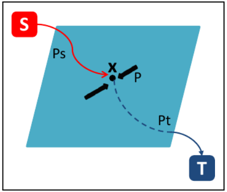


$$\xRightarrow{\Gamma}$$



- Anisotropic TV: sub-modular,

Gamma convergence

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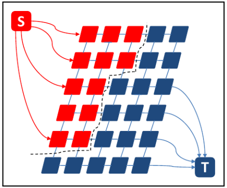
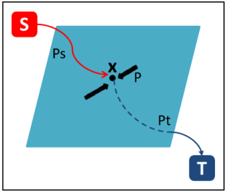


$$\overset{\Gamma}{\Longrightarrow}$$



- Anisotropic TV: sub-modular, Can use standard graph cut methods.
- Isotropic TV:

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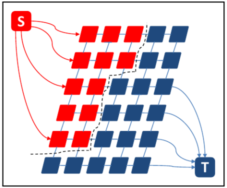
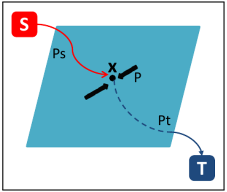


$$\overset{\Gamma}{\Longrightarrow}$$



- Anisotropic TV: sub-modular, Can use standard graph cut methods.
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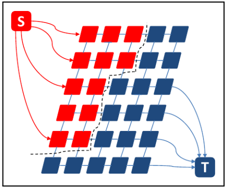
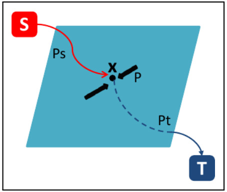


$$\overset{\Gamma}{\Longrightarrow}$$



- Anisotropic TV: sub-modular, Can use standard graph cut methods.
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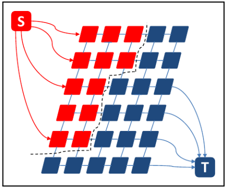
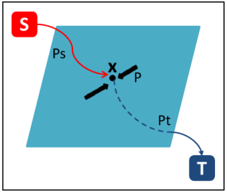


$$\overset{\Gamma}{\Longrightarrow}$$



- Anisotropic TV: sub-modular, Can use standard graph cut methods.
- Isotropic TV: not sub-modular, Cannot use stander graph cut methods, primal-dual approach is faster even !

Literature: Gamma convergence for discrete TV

- Isotropic TV: $h \mapsto 0$ $h =$ mesh size.
  Braides-2002, Chambolle-2004,
  Chambolle-Caselles-Novaga-Cremers-Pock-2010,
- Anisotropic TV: $h \mapsto 0$ $h =$ mesh size.
  Chambolle-Caselles-Novaga-Cremers-Pock-2010,
  Gennip-Bertozzi-2012, Trillo-Slepcev-2013,

Multiphase Approaches

From Two-phase to multi-phases

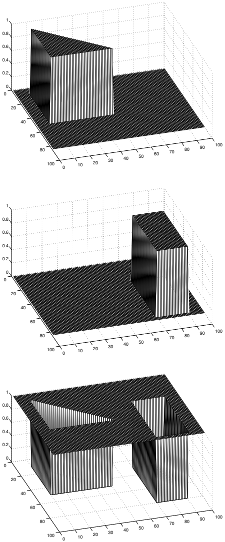
- Related to garph cut, α-expansion and $\alpha - \beta$ swap are mostly popular approaches for multiphase "labelling".
- Approximations are made and upper bounded has been given.

Ref: Y. Boykov and O. Veksler and R. Zabih: Fast approximate energy minimization via graph cuts, IEEE Transactions on Pattern Analysis and Machine Intelligence, 23, 1222-1239, (2001).

Multiphase problems – Approach I

We need to identify n characteristic functions $\psi_i(x),\ \ i = 1, 2 \cdots n$:

$$\psi_i(x) \in \{0, 1\}, \quad \sum_{i=1}^{n} \psi_i(x) = 1.$$



Multiphase problems – Approach II

Each point $x \in \Omega$ is labelled by a vector function:

$$u(x) = (u_1(2), u_2(x), \cdots u_d(x)).$$

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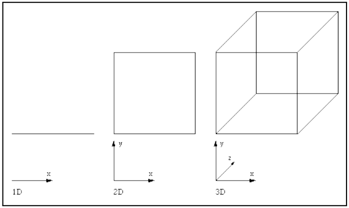
- Multiphase: Total number of phases $n = 2^d$. (Chan-Vese)



$$u_i(x) \in \{0, 1\}.$$

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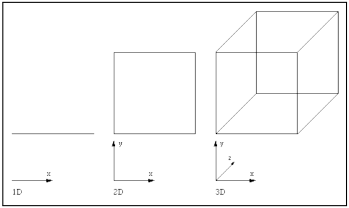


$$u_i(x) \in \{0, 1\}.$$

- More than binary labels: Total number of phases $n = B^d$.

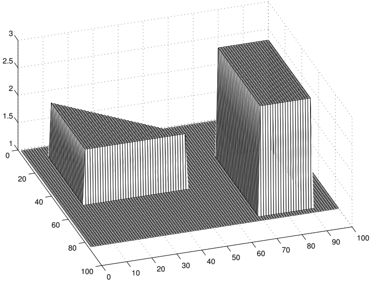
$$u_i(x) \in \{0, 1, 2, \cdots B\}.$$

Multiphase problems – Approach III

Each point $x \in \Omega$ is labelled by

$$u(x) = i, \quad i = 1, 2, \cdots n.$$

- One label function is enough for any n phases.
- More generall $u(x) = \ell_i, \; i = 1, 2, \cdots n.$



Literature: Multiphase Approach I

Zach-et-al-2008 (VMV),
Lellmann-Kappes-Yuan-Becker-Schnörr-2009 (SSVM),
Lellman-Schnorr-2011 (SIIMS), Li-Ng-Zeng-Chen-2010 (SIIMS),
Lellman-Lellman-Widman-Schnorr-2013 (IJCV),
Qiao-Wang-Ng-2013, Bae-Yuan-Tai-2011 (IJCV)

Literature: Multiphase Approach II

Vese-Chan-2002 (IJCV),Lie-Lysaker-Tai-2006 (IEEE TIP), Brown-Chan-Bresson-2010 (cam-report 10-43), Bae-Tai 2009/2014 (JMIV)

Literature: Multiphase Approach III

Chung-Vese-2005 (EMMCVPR), Lie-Lysaker-Tai-2006 (Math. Comp), Ishikawa-2004 (PAMI), Pock-Chambolle-Bischof-Cremers 2008/2010 (SIIMS), Kim-Kang-2012 (IEEE TIP), Jung-Kang-Shen-2007, Wei-Wang-2009, Luo-Tong-Luo-Wei-Wang-2009, Bae-Yuan-Tai-Boykov 2010/2014

Multiphase Approach (I)

Graph for characteristic functions

Ref: Yuan-Bae-T.-Boykov (ECCV10): A continuous max-flow approach to Potts model, Computer Vision–ECCV (2010), pp. 379–392.

Ref: Bae-Yuan-Tai: Global minimization for continuous multiphase partitioning problems using a dual approach, International journal of computer vision, 92, 112–129(2011).

Multi-partitioning problem

Multi-partitioning problem (Pott's model)

$$\min_{\{\Omega_i\}} \sum_{i=1}^{n} \int_{\Omega_i} f_i dx + \sum_{i=1}^{n} \int_{\partial\Omega_i} g(x) ds,$$

$$\text{such that} \quad \cup_{i=1}^{n} \Omega_i = \Omega, \quad \cap_{i=1}^{n} \Omega_i = \emptyset$$

Multi-partitioning problem (Pott's model)

$$\min_{\{\Omega_i\}} \sum_{i=1}^{n} \int_{\Omega_i} f_i \, dx + \sum_{i=1}^{n} \int_{\partial\Omega_i} g(x) ds,$$

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Pott's model in terms of characteristic functions

$$\min_{u_i(x)\in\{0,1\}} \sum_{i=1}^{n} \int_{\Omega} u_i(x) f_i(x)\, dx + \sum_{i=1}^{n} \int_{\Omega} g(x) \left|\nabla u_i\right| \, dx\,, \;\text{ s.t. } \sum_{i=1}^{n} u_i(x) = 1$$

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$$u_i(x) \; = \; \chi_{\Omega_i}(x) \; := \left\{ \begin{array}{ll} 1, & x \in \Omega_i \\ 0, & x \notin \Omega_i \end{array} \right. , \quad i = 1, \ldots, n$$

A convex relaxation approach

Relaxed Potts' model in terms of characteristic functions (primal model)

$$\min_{u} \; E^P(u) = \sum_{i=1}^{n} \int_{\Omega} u_i(x) f_i(x)\, dx + \sum_{i=1}^{n} \int_{\Omega} g(x)\, |\nabla u_i| \; dx\,,$$

A convex relaxation approach

Relaxed Potts' model in terms of characteristic functions (primal model)

$$\min_u \; E^P(u) = \sum_{i=1}^n \int_\Omega u_i(x) f_i(x)\, dx + \sum_{i=1}^n \int_\Omega g(x)\, |\nabla u_i| \; dx \,,$$

$$\text{s.t. } \; u \in \triangle_+ = \{(u_1(x), \dots, u_n(x)) \,|\, \sum_{i=1}^n u_i(x) \; = \; 1 \,; \quad u_i(x) \geq 0 \,\}$$

- Convex optimization problem
- Optimization techniques: Zach et. al. alternating TV minimization. Lellmann et. al: Douglas Rachford splitting and special thresholding, Bae-Yuan-T. (2010), Chambolle-Crmer-Pock (2012).

Dual formulation of relaxation: Bae-Yuan-T. (IJCV, 2011)

Dual model: $C_\lambda := \{p : \Omega \mapsto \mathbb{R}^2 \,|\, |p(x)|_2 \leq g(x)\,,\ p_n|_{\partial\Omega} = 0\,\}\,,$

- Hence the primal-dual model can be optimized pointwise for u

$$\min_{u \in \triangle_+} \sum_{i=1}^{n} \int_\Omega u_i(x) f_i(x)\, dx + \sum_{i=1}^{n} \int_\Omega g(x)\, |\nabla u_i|\ dx\,,$$

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$$\max_{p_i \in C_\lambda} \min_{u \in \triangle_+} E(u,p) = \int_\Omega \sum_{i=1}^n u_i (f_i + \operatorname{div} p_i) \, dx$$

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$$= \max_{p_i \in C_\lambda} \int_\Omega \min_{u(x) \in \triangle_+} \sum_{i=1}^n u_i(x)(f_i(x) + \operatorname{div} p_i(x))\, dx$$

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$$= \max_{p_i \in C_\lambda} \int_\Omega \big\{ \min(f_1 + \operatorname{div} p_1, \ldots, f_n + \operatorname{div} p_n) \big\}\, dx$$

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$$= \max_{p_i \in C_\lambda} E^D(p)$$

Convex relaxation over unit simplex

$$p^* \in \arg \sup_{p_i \in C_\alpha} E^D(p) = \int_\Omega \min(f_1 + \operatorname{div} p_1, \ldots, f_n + \operatorname{div} p_n)\, dx$$

$$u^* \in \arg \min_{u \in \triangle_+} E(u, p^*) = \int_\Omega \min_{u(x) \in \triangle_+} \sum_{i=1}^n u_i(x)(f_i(x) + \operatorname{div} p_i^*(x))\, dx$$

Theorem

Let p^* be optimal to the dual model. For each $x \in \Omega$, a binary primal optimal variable $u^*(x)$ can be recovered by

$$u_k^*(x) \;=\; \begin{cases} 1 & \text{if } k = \arg\min_{i=1,\ldots,n} (f_i(x) + \operatorname{div} p_i^*(x)) \\ 0 & \text{otherwise} \end{cases} ,$$

provided the n values $(f_1(x) + \operatorname{div} p_1^*(x), ..., f_n(x) + \operatorname{div} p_n^*(x))$ have a unique minimizer. Then (u^*, p^*) is a saddle point, i.e.

$$E^P(u^*) = E(u^*, p^*) = E^D(p^*)$$

Convex relaxation over unit simplex

$$p^* \in \arg\sup_{p_i \in C_\alpha} E^D(p) = \int_\Omega \min(f_1 + \operatorname{div} p_1, \ldots, f_n + \operatorname{div} p_n)\, dx$$

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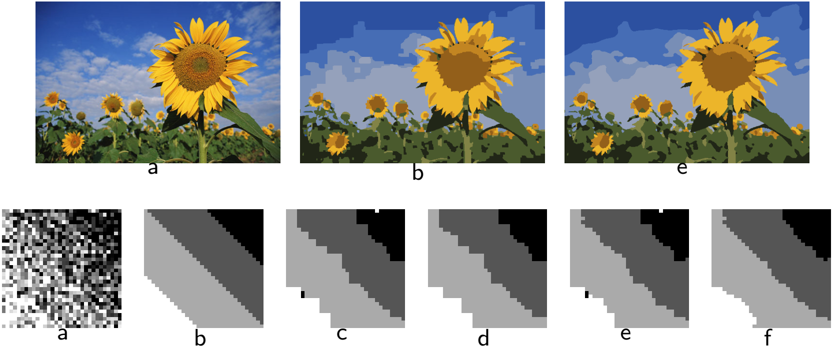
Theorem

Let p^* be optimal to the dual model. If the n values $(f_1(x) + \operatorname{div} p_1^*(x), \ldots, f_n(x) + \operatorname{div} p_n^*(x))$ have at most two minimizers for each $x \in \Omega$, there exists optimal binary primal variables u^* such that (u^*, p^*) is a saddle point, i.e.

$$E^P(u^*) = E(u^*, p^*) = E^D(p^*)$$

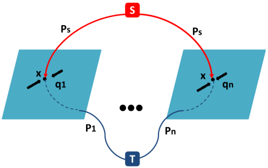
Convex relaxation over unit simplex



Top: (a) Input, (b) alpha expansion (c) dual model.
Bottom: (d) Input, (e) ground truth, (f) alpha expansion, (g) alpha-beta swap, (h) Lellmann et. al., (i) dual model.

Multiple Phases: Convex Relaxed Potts Model (CR-PM) –Yuan-Bae-T.-Boykov (ECCV'10)

Continuous Max-Flow Model (CMF-PM)



1. n copies Ω_i, $i = 1, \ldots, n$, of Ω;
2. For $\forall x \in \Omega$, the same source flow $p_s(x)$ from the source s to $x \in \Omega_i$, $i = 1, \ldots, n$, simultaneously;
3. For $\forall x \in \Omega$, the sink flow $p_i(x)$ from x at Ω_i, $i = 1, \ldots, n$, of Ω to the sink t. $p_i(x)$, $i = 1, \ldots, n$, may be different one by one;
4. The spatial flow $q_i(x)$, $i = 1, \ldots, n$ defined within each Ω_i.

Max-flow on this graph

Max-Flow:

$$\max_{p_s,p,q}\{P(p_s,p,q)=\int_\Omega p_s dx\}$$
$$|q_i(x)|\le g(x), \quad p_i(x)\le f_i(x),$$
$$(\mathrm{div} q_i - p_s + p_i)(x)=0,\ \ i=1,2,\cdots n.$$

Note that

$$p_s(x)=\mathrm{div} q_i(x)+p_i(x), i=1,2\cdots n.$$

Thus

$$p_s(x)=\min(f_1+\mathrm{div}\, p_1,\ldots,f_n+\mathrm{div}\, p_n).$$

Therefore, the maximum of $\int_\Omega p_s(x)$ is:

$$\max_{|q_i(x)|\le g(x)}\int_\Omega \min(f_1+\mathrm{div}\, p_1,\ldots,f_n+\mathrm{div}\, p_n)dx$$

(Convex) min-cut on this graph

$$\max_{p_s,p,q} \min_u \{ E(p_s,p,q,u) = \int_\Omega p_s dx + \sum_{i=1}^m u_i(\mathrm{div} q_i - p_s + p_i) dx \}$$

$$\text{s.t. } p_i(x) \le f_i(x), \quad |q_i(x)| \le g(x).$$

Rearranging the energy functional $E(\cdot)$, we that

$$E(p_s,p,q,u) = \int_\Omega (1 - \sum_{i=1}^m u_i)p_s + \sum_{i=1}^m u_i p_i + \sum_{i=1}^m u_i \mathrm{div} q_i . dx.$$

The following constraint are automatically satisfied from the optimization:

$$u_i(x) \ge 0, \sum_{i=1}^m u_i = 1.$$

(Convex) min-cut: Dual formulation

It gives the convex min-cut from the dual formulation:

$$\min_{u_i} \int_\Omega u_i(x) f_i(x) + g(x) |\nabla u_i(x)|$$

$$\text{s.t} \sum_{i=1}^{n} u_i(x) = 1, \;\; u_i(x) \geq 0.$$

Algorithms

Augmented Lagrangian functional

$$\int_\Omega p_s \, dx + \sum_{i=1}^n \langle u_i, \operatorname{div} q_i - p_s + p_i \rangle - \frac{c}{2} \sum_{i=1}^n \|\operatorname{div} q_i - p_s + p_i\|^2$$

Augmented Lagrangian Method (ADMM):
Initialize p_s^0, p_i^0, q^0 and ϕ^0. For $k = 0, 1, ...$

$$q_i^{k+1} := \arg \max_{\|q_i\|_\infty \leq \alpha} -\frac{c}{2} \left\| \operatorname{div} q_i + p_i^k - p_s^k - u_i^k / c \right\|^2, \; i = 1, ..., n$$

$$p_i^{k+1} := \arg \max_{p_i(x) \leq \rho(l_i, x)} -\frac{c}{2} \left\| p_i + \operatorname{div} q_i^{k+1} - p_s^k - u_i^k / c \right\|^2, \; i = 1, ..., n$$

$$p_s^{k+1} := \arg \max_{p_s} \int_\Omega p_s \, dx - \frac{c}{2} \sum_{i=1}^n \left\| p_s - (p_i^{k+1} + \operatorname{div} q_i^{k+1}) + u_i^k / c \right\|^2,$$

$$u_i^{k+1} = u_i^k - c \, (\operatorname{div} q_i^{k+1} - p_s^{k+1} + p_i^{k+1}), \; i = 1, ..., n$$

Algorithms

Comparisons between algorithms: Zach et al 08, Lellmann et al. 09 and the proposed max-flow algorithm: for three images, different precision ϵ are used and the total number of iterations to reach convergence is evaluated.

	Brain $\epsilon \leq 10^{-5}$	Flower $\epsilon \leq 10^{-4}$	Bear $\epsilon \leq 10^{-4}$
Zach et al 08	fail to reach such a precision		
Lellmann et al. 09	421 iter.	580 iter.	535 iter.
Proposed algorithm	88 **iter.**	147 **iter.**	133 **iter.**

Outline of this presentation

First part: Exact optimization

- Will focus on two approaches for multiphase problems with global optimality guarantee.
- Both can be formulated as max-flow/min-cut problems on a graph in discrete setting.
- Both can be exactly formulated as convex problems on continuous setting. Dual problems can be formulated as continuous max-flow problems.

Second part: Approximate optimization

- Convex relaxations for broader set of non-convex problems.
- Includes Potts' model and joint optimization of regions and region parameters in image segmentation.
- Dual problems can be formulated as max-flow, but now there may be a duality gap to original problems

Problem formulations

Image partition problems with multiple regions
Given input image I^0 defined over Ω. Find partition $\{\Omega_i\}_{i=1}^n$ of Ω by solving

$$\min_{\{\Omega_i\}_{i=1}^n} \sum_{i=1}^n \int_{\Omega_i} f_i(I^0(x))\,dx + \alpha R(\{\partial\Omega_i\}_{i=1}^n)$$

$$\text{such that} \quad \cup_{i=1}^n \Omega_i = \Omega, \quad \cap_{i=1}^n \Omega_i = \emptyset$$

n is known or unknown in advance. Example (Potts' model):

$$\min_{\{\Omega_i\}_{i=1}^n} \sum_{i=1}^n \int_{\Omega_i} f_i(I^0(x))\,dx + \sum_{i=1}^n \alpha \int_{\partial\Omega_i} ds,$$

Discretized problem is NP-hard for $n > 2$

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n is known or unknown in advance. Example (Potts' model):

$$\min_{\{\Omega_i\}_{i=1}^n} \sum_{i=1}^n \int_{\Omega_i} |I^0(x) - \xi_i|^\beta\, dx + \sum_{i=1}^n \alpha \int_{\partial\Omega_i} ds,$$

Discretized problem is NP-hard for $n > 2$

Problem formulations

Image partition problems with multiple regions
Given input image I^0 defined over Ω. Find partition $\{\Omega_i\}_{i=1}^n$ of Ω by solving

$$\min_{\{\Omega_i\}_{i=1}^n, \{\xi_i\}_{i=1}^n \in X} \sum_{i=1}^n \int_{\Omega_i} f_i(\xi_i, I^0(x))\, dx + \alpha R(\{\partial\Omega_i\}_{i=1}^n)$$

$$\text{such that} \quad \cup_{i=1}^n \Omega_i = \Omega, \quad \cap_{i=1}^n \Omega_i = \emptyset$$

n is known or unknown in advance. Example:

$$\min_{\{\Omega_i\}_{i=1}^n, \{\xi_i\}_{i=1}^n \in \mathbb{R}} \sum_{i=1}^n \int_{\Omega_i} |I^0(x) - \xi_i|^\beta\, dx + \sum_{i=1}^n \alpha \int_{\partial\Omega_i} ds,$$

If regularization $\alpha = 0$: "k-mean" problem, which is known to be NP-hard.

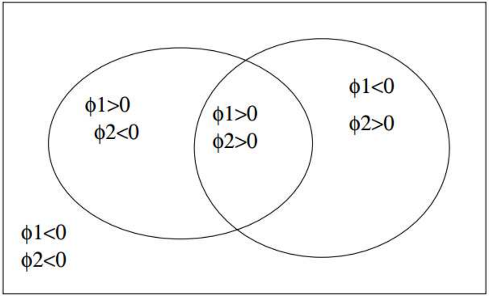
Different representations of partitions in terms of functions

- 1) Vector function: $\mathbf{u}(x) = (u^1(x), ..., u^n(x)) = e_i$ for $x \in \Omega_i$
- 2) Labeling function: $\ell(x) = i$ for all $x \in \Omega_i$
- 3) log representation by $m = \log_2(n)$ binary functions $\phi^1, ...\phi^m$

$x \in \Omega_1$ iff	$\mathbf{u}(x) = e_1$	$\ell(x) = 1$	$\phi^1(x) = 1, \phi^2(x) = 0$
$x \in \Omega_2$ iff	$\mathbf{u}(x) = e_2$	$\ell(x) = 2$	$\phi^1(x) = 1, \phi^2(x) = 1$
$x \in \Omega_3$ iff	$\mathbf{u}(x) = e_3$	$\ell(x) = 3$	$\phi^1(x) = 0, \phi^2(x) = 0$
$x \in \Omega_4$ iff	$\mathbf{u}(x) = e_4$	$\ell(x) = 4$	$\phi^1(x) = 0, \phi^2(x) = 1$

Table: Representation of 4 regions.

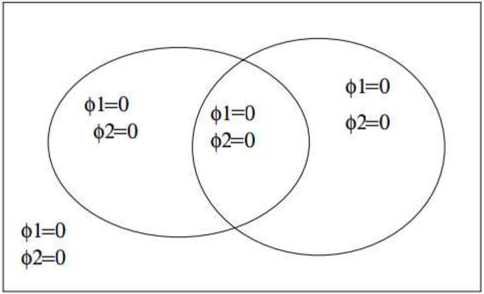
Log representation by two binary functions



$$\begin{aligned}
\Omega_1 &= \{x \in \Omega \,\text{s.t.}\ \phi^1(x) > 0, \phi^2(x) < 0\} \\
\Omega_2 &= \{x \in \Omega \,\text{s.t.}\ \phi^1(x) > 0, \phi^2(x) > 0\} \\
\Omega_3 &= \{x \in \Omega \,\text{s.t.}\ \phi^1(x) < 0, \phi^2(x) < 0\} \\
\Omega_4 &= \{x \in \Omega \,\text{s.t.}\ \phi^1(x) < 0, \phi^2(x) > 1\}
\end{aligned}$$

Vese and Chan 2002, A Multiphase Level Set Framework for Image Segmentation Using the Mumford and Shah Model, International Journal of Computer Vision 50(3), 271–293

Log representation by two binary functions



$$\begin{aligned}
\Omega_1 &= \{x \in \Omega \,\text{s.t. } \phi^1(x) = 1, \phi^2(x) = 0\} \\
\Omega_2 &= \{x \in \Omega \,\text{s.t. } \phi^1(x) = 1, \phi^2(x) = 1\} \\
\Omega_3 &= \{x \in \Omega \,\text{s.t. } \phi^1(x) = 0, \phi^2(x) = 0\} \\
\Omega_4 &= \{x \in \Omega \,\text{s.t. } \phi^1(x) = 0, \phi^2(x) = 1\}
\end{aligned}$$

Lie et al. 2006, A Binary Level Set Model and Some Applications to Mumford–Shah Image Segmentation, IEEE transactions on image processing, 15(5), pg. 1171 - 1181

Log representation by two binary functions

4 regions as intersection of 2 level set functions

$$\min_{\phi^1,\phi^2} \ \alpha \int_\Omega |\nabla H(\phi^1)| + \alpha \int_\Omega |\nabla H(\phi^2)|$$
$$+ \int_\Omega \{H(\phi^1)H(\phi^2)f_2 + H(\phi^1)(1 - H(\phi^2))f_1$$
$$+(1 - H(\phi^1))H(\phi^2)f_4 + (1 - H(\phi^1))(1 - H(\phi^2))f_3\}dx.$$

- Heaviside function $H(\phi) = 1$ if $\phi > 0$ and $H(\phi) = 0$ if $\phi < 0$
- Interpretation of regions:

$$\begin{array}{l}
\Omega_1 = \{x \in \Omega \,\text{s.t.}\ \phi^1(x) > 0, \phi^2(x) < 0\} \\
\Omega_2 = \{x \in \Omega \,\text{s.t.}\ \phi^1(x) > 0, \phi^2(x) > 0\} \\
\Omega_3 = \{x \in \Omega \,\text{s.t.}\ \phi^1(x) < 0, \phi^2(x) < 0\} \\
\Omega_4 = \{x \in \Omega \,\text{s.t.}\ \phi^1(x) < 0, \phi^2(x) < 0\}
\end{array}$$

Vese and Chan 2002, A Multiphase Level Set Framework for Image Segmentation Using the Mumford and Shah Model, International Journal of Computer Vision 50(3), 271–293

Log representation by two binary functions

4 **regions as intersection of** 2 **binary functions**

$$\min_{\phi^1,\phi^2} \alpha \int_\Omega |\nabla \phi^1| + \alpha \int_\Omega |\nabla \phi^2| +$$
$$+ \int_\Omega \{\phi^1 \phi^2 f_2 + \phi^1(1-\phi^2) f_1$$
$$+(1-\phi^1)\phi^2 f_4 + (1-\phi^1)(1-\phi^2) f_3\} dx.$$

- Minimize over constraint $\phi^1(x), \phi^2(x) \in \{0,1\} \ \forall \ x \in \Omega$.
- Interpretation of regions:

$$\boxed{\begin{aligned}
\Omega_1 &= \{x \in \Omega \,\text{s.t.}\ \phi^1(x) = 1, \phi^2(x) = 0\} \\
\Omega_2 &= \{x \in \Omega \,\text{s.t.}\ \phi^1(x) = 1, \phi^2(x) = 1\} \\
\Omega_3 &= \{x \in \Omega \,\text{s.t.}\ \phi^1(x) = 0, \phi^2(x) = 0\} \\
\Omega_4 &= \{x \in \Omega \,\text{s.t.}\ \phi^1(x) = 0, \phi^2(x) = 1\}
\end{aligned}}$$

Lie et al. 2006, A Binary Level Set Model and Some Applications to Mumford–Shah Image Segmentation, IEEE transactions on image processing, 15(5), pg. 1171 - 1181

Convex formulation log representation

$$\min_{\phi^1,\phi^2\in\{0,1\}} \alpha \int_\Omega |\nabla \phi^1| + \alpha \int_\Omega |\nabla \phi^2|$$

$$\int_\Omega (1-\phi^1(x))C(x) + (1-\phi^2(x))D(x) + \phi^1(x)A(x) + \phi^2(x)B(x)\,dx$$

$$+ \int_\Omega \max\{\phi^1(x)-\phi^2(x),0\}E(x) - \min\{\phi^1(x)-\phi^2(x),0\}F(x)\,dx$$

$$\begin{cases} A(x)+B(x) & = f_2(x) \\ C(x)+D(x) & = f_3(x) \\ A(x)+E(x)+D(x) & = f_1(x) \\ B(x)+F(x)+C(x) & = f_4(x) \end{cases}$$

- Energy is convex provided $E(x), F(x) \geq 0$ for all $x \in \Omega$.
- Discrete counterpart is submodular iff $\exists\, E(x), F(x) \geq 0$ for all $x \in \Omega$ (otherwise NP-hard)

Bae and Tai, Efficient Global Minimization Methods for Image Segmentation Models with Four Regions, Journal of Mathematical Imaging and Vision, 2014

Convex formulation log representation

$$\min_{\phi^1(x),\phi^2(x)\in[0,1]} \alpha \int_\Omega |\nabla \phi^1| + \alpha \int_\Omega |\nabla \phi^2|$$

$$\int_\Omega (1-\phi^1(x))C(x) + (1-\phi^2(x))D(x) + \phi^1(x)A(x) + \phi^2(x)B(x)\, dx$$

$$+ \int_\Omega \max\{\phi^1(x) - \phi^2(x), 0\}E(x) - \min\{\phi^1(x) - \phi^2(x), 0\}F(x)\, dx$$

$$\begin{cases} A(x) + B(x) & = f_2(x) \\ C(x) + D(x) & = f_3(x) \\ A(x) + E(x) + D(x) & = f_1(x) \\ B(x) + F(x) + C(x) & = f_4(x) \end{cases}$$

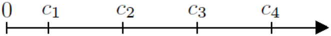
- Minimize over convex constraint $\phi^1(x), \phi^2(x) \in [0,1]$ $\forall x \in \Omega$.
- **Theorem**: Binary functions obtained by thresholding solution of convex problem ϕ^1, ϕ^2 at any level $t \in (0,1]$ is a **global** minimizer to the original problem.

Convex formulation log representation

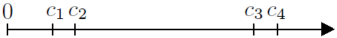
- Exists $E(x), F(x) \geq 0$ if $f_2(x) + f_3(x) \leq f_1(x) + f_4(x)$.
- In case of $f_i = |I^0 - c_i|^\beta$, a sufficient condition is

$$|c_2 - I|^\beta + |c_3 - I|^\beta \leq |c_1 - I|^\beta + |c_4 - I|^\beta, \qquad \forall\, I \in [0, L],$$

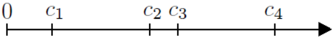
- Proposition 1: Let $0 \leq c_1 < c_2 < c_3 < c_4$. Condition is satisfied for all $I \in [\frac{c_2 - c_1}{2}, \frac{c_4 - c_3}{2}]$ for any $\beta \geq 1$.
- Proposition 2: Let $0 \leq c_1 < c_2 < c_3 < c_4$. There exists a $\mathcal{B} \in \mathbb{N}$ such that condition is satisfied for any $\beta \geq \mathcal{B}$.



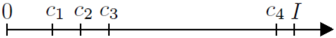
Condition satisfied



Condition satisfied



Condition satisfied



Condition not satisfied for $\beta = 1$

Convex formulation log representation

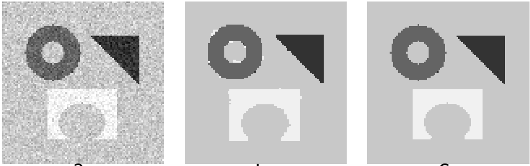


Figure: L^2 data fidelity: (a) input, (b) level set method gradient descent, (c) New convex formulation of Chan-Vese model (global minimum).

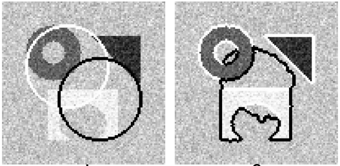


Figure: Level set method: (d) bad initialization, (e) result.

Convex formulation log representation

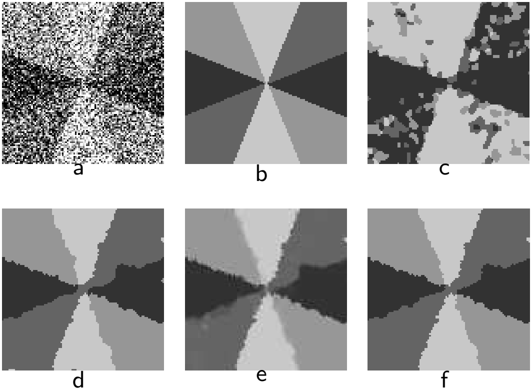


Figure: (a) Input image, (b) ground truth, (c) level set method gradient descent, (d) global minimum computed by new graph cut approach in discrete setting, (e) New convex optimization approach in continuous setting before threshold, (f) convex minimization approach after threshold (global optimum).

Convex formulation log representation

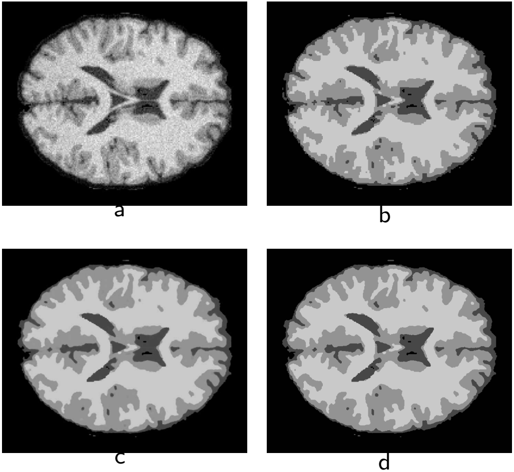


Figure: L^2 data fidelity: (a) Input, (b) global minimum discrete Chan-Vese model 4 neighbors, (c) convex formulation before threshold, (d) convex formulation after threshold (global minimum).

Convex formulation log representation

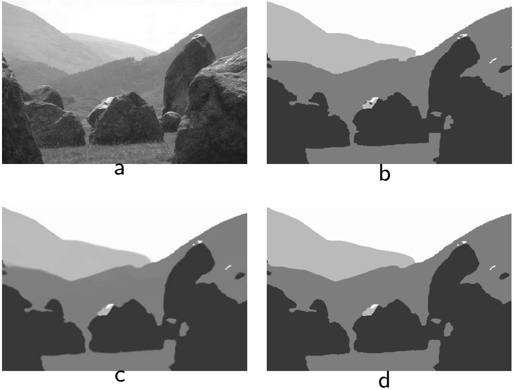


Figure: Segmentation with L^2 data term: (a) Input, (b) graph cut 4 neighbors (c) convex formulation before threshold, (d) convex formulation after threshold (global minimum).

Convex formulation log representation

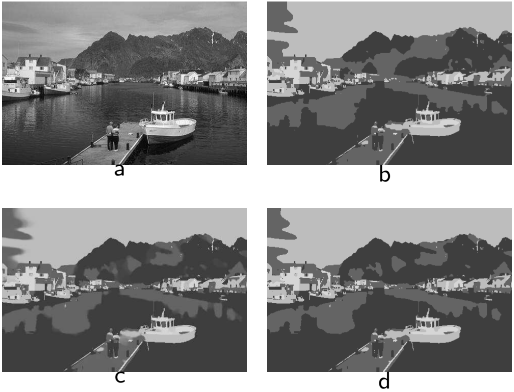


Figure: Segmentation with L^2 data term: (a) Input, (b) result graph cut 8 neighbors in discrete setting (c) result convex formulation before threshold, (d) result convex formulation after threshold (global optimum).

Convex formulation log representation

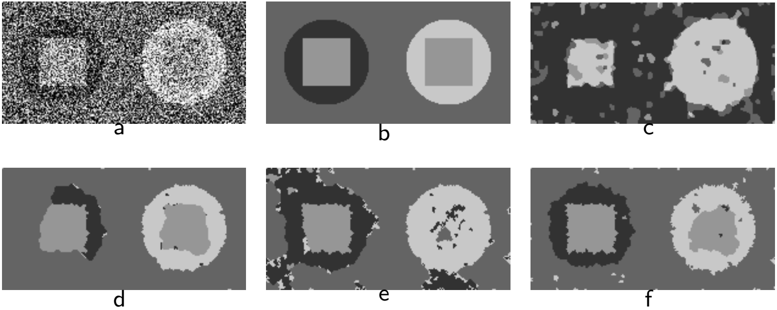


Figure: (a) Input image, (b) ground truth, (c) gradient descent, (d) alpha expansion, (e) alpha-beta swap, (f) convex model.

Discrete energy, anisotropic TV

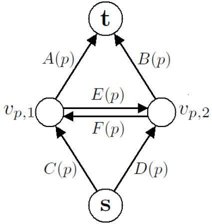
$$\min_{\phi^1,\phi^2\in\mathcal{B}} E_d(\phi^1,\phi^2) = \sum_{p\in\mathcal{P}} E_p^{data}(\phi_p^1,\phi_p^2)$$

$$+\alpha \sum_{p\in\mathcal{P}} \sum_{q\in\mathcal{N}_p^k} w_{pq}|\phi_p^1-\phi_q^1| + \alpha \sum_{p\in\mathcal{P}} \sum_{q\in\mathcal{N}_p^k} w_{pq}|\phi_p^2-\phi_q^2|$$

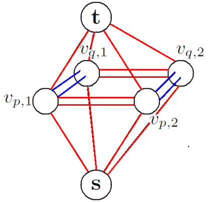
$$E_p^{data}(\phi_p^1,\phi_p^2) = \{\phi_p^1\phi_p^2 f_2(p) + \phi_p^1(1-\phi_p^2)f_1(p))$$
$$+(1-\phi_p^1)\phi_p^2 f_4(p) + (1-\phi_p^1)(1-\phi_p^2)f_3(p)\}.$$

Log representation - minimization by graph cuts

Graph construction



1 grid point



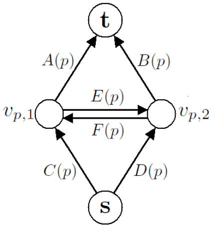
2 grid points

- Associate two vertices to each grid point ($v_{p,1}$ and $v_{p,2}$)
- For any cut (V_s, V_t)
  - If $v_{p,i} \in V_s$ then $\phi^i = 1$ for $i = 1, 2$
  - If $v_{p,i} \in V_t$ then $\phi^i = 0$ for $i = 1, 2$
- Figure left: graph corresponding to one grid point p
- Figure right: graph corresponding to two grid points p and q
  - Red: Data edges, constituting $E^{data}(\phi_1, \phi_2)$
  - Blue: Regularization edges with weight w_{pq}

Bae and Tai EMMCVPR 2009, Kolmogorov PAMI 2004

Log representation - minimization by graph cuts

Graph construction



- Linear system for finding edge weights

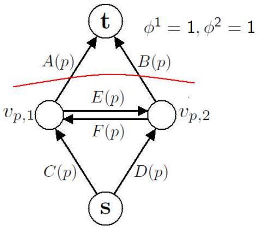
$$\begin{cases} A(p)+B(p) & = f_2(p) \\ C(p)+D(p) & = f_3(p) \\ A(p)+E(p)+D(p) & = f_1(p) \\ B(p)+F(p)+C(p) & = f_4(p) \end{cases}$$

$$\text{such that } E(p), F(p) \geq 0$$

- For each p, $E_p^{\text{data}}(\phi_p^1, \phi_p^2)$ interaction between two binary variables. Linear system has solution iff $E_p^{\text{data}}(\phi_p^1, \phi_p^2)$ is submodular.

Log representation - minimization by graph cuts

Graph construction



- Linear system for finding edge weights

$$\begin{cases} A(p) + B(p) & = f_2(p) \\ C(p) + D(p) & = f_3(p) \\ A(p) + E(p) + D(p) & = f_1(p) \\ B(p) + F(p) + C(p) & = f_4(p) \end{cases}$$

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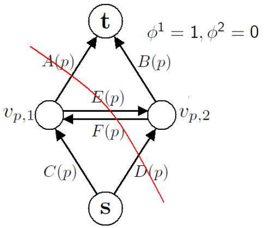
Graph construction



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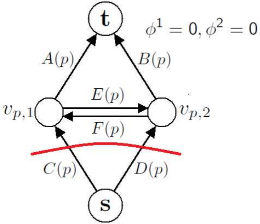
Graph construction



- Linear system for finding edge weights

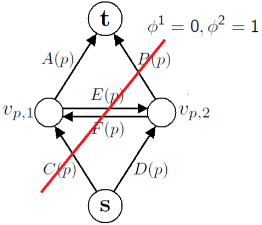
$$\begin{cases} A(p) + B(p) & = f_2(p) \\ C(p) + D(p) & = f_3(p) \\ A(p) + E(p) + D(p) & = f_1(p) \\ B(p) + F(p) + C(p) & = f_4(p) \end{cases}$$

$$\text{such that } E(p), F(p) \geq 0$$

- For each p, $E_p^{\text{data}}(\phi_p^1, \phi_p^2)$ interaction between two binary variables. Linear system has solution iff $E_p^{\text{data}}(\phi_p^1, \phi_p^2)$ is submodular.

Log representation - minimization by graph cuts

Graph construction



- Linear system for finding edge weights

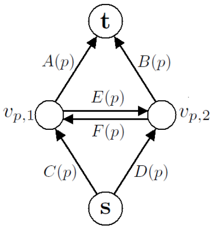
$$\begin{cases} A(p) + B(p) & = f_2(p) \\ C(p) + D(p) & = f_3(p) \\ A(p) + E(p) + D(p) & = f_1(p) \\ B(p) + F(p) + C(p) & = f_4(p) \end{cases}$$

$$\text{such that } E(p), F(p) \geq 0$$

- For each p, $E_p^{\text{data}}(\phi_p^1, \phi_p^2)$ interaction between two binary variables. Linear system has solution iff $E_p^{\text{data}}(\phi_p^1, \phi_p^2)$ is submodular.

Log representation - minimization by graph cuts

Graph construction



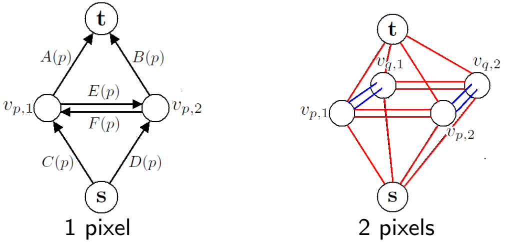
- Linear system for finding edge weights

$$\begin{cases} A(p) + B(p) & = f_2(p) + \sigma(p) \\ C(p) + D(p) & = f_3(p) + \sigma(p) \\ A(p) + E(p) + D(p) & = f_1(p) + \sigma(p) \\ B(p) + F(p) + C(p) & = f_4(p) + \sigma(p) \end{cases}$$

$$\text{such that } E(p), F(p) \geq 0$$

- For each p, $E_p^{\text{data}}(\phi_p^1, \phi_p^2)$ interaction between two binary variables. Linear system has solution iff $E_p^{\text{data}}(\phi_p^1, \phi_p^2)$ is submodular.

Dual max-flow problem over graph



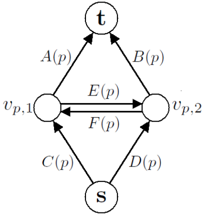
Max-flow problem

$$\sup_{p_s, p_t, p^{12}, q} \int_\Omega p_s^1(x) + p_s^2(x)\, dx$$

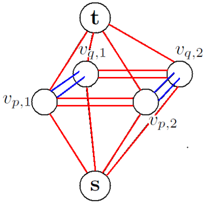
subject to

$$p_s^1(x) \leq C(x), \quad p_s^2(x) \leq D(x), \quad p_t^1(x) \leq A(x), \quad p_t^2 \leq B(x),$$
$$-F(x) \leq p^{12}(x) \leq E(x), \quad |q^1(x)|_1 \leq \alpha, \quad |q^2(x)|_1 \leq \alpha, \quad \forall\, x \in \Omega.$$
$$\operatorname{div} q^1(x) - p_s^1(x) + p_t^1(x) + p^{12}(x) = 0, \quad \forall\, x \in \Omega$$
$$\operatorname{div} q^2(x) - p_s^2(x) + p_t^2(x) - p^{12}(x) = 0, \quad \forall\, x \in \Omega.$$

Continuous generalization of max-flow problem



1 pixel



2 pixels

Dual formulation

$$\sup_{p_s, p_t, p^{12}, q} \int_\Omega p_s^1(x) + p_s^2(x)\, dx$$

subject to

$$p_s^1(x) \leq C(x), \quad p_s^2(x) \leq D(x), \quad p_t^1(x) \leq A(x), \quad p_t^2 \leq B(x),$$

$$-F(x) \leq p^{12}(x) \leq E(x), \quad |q^1(x)|_2 \leq \alpha, \quad |q^2(x)|_2 \leq \alpha, \quad \text{a.e. } x \in \Omega.$$

$$\operatorname{div} q^1(x) - p_s^1(x) + p_t^1(x) + p^{12}(x) = 0, \quad \text{a.e. } x \in \Omega$$

$$\operatorname{div} q^2(x) - p_s^2(x) + p_t^2(x) - p^{12}(x) = 0, \quad \text{a.e. } x \in \Omega.$$

Continuous generalization of max-flow problem

Primal-dual formulation

$$\inf_{\phi^1,\phi^2} \sup_{p_s,p_t,p^{12},q} \int_\Omega p_s^1(x) + p_s^2(x)\,dx$$

$$+ \int_\Omega \phi^1(x)(\operatorname{div} q^1(x) - p_s^1(x) + p_t^1(x) + p^{12}(x))\,dx$$

$$+ \int_\Omega \phi^2(x)(\operatorname{div} q^2(x) - p_s^2(x) + p_t^2(x) - p^{12}(x))\,dx\},$$

subject to

$$p_s^1(x) \le C(x), \;\; p_s^2(x) \le D(x), \;\; p_t^1(x) \le A(x), \;\; p_t^2 \le B(x),$$
$$- F(x) \le p^{12}(x) \le E(x), \;\; |q^1(x)|_2 \le \alpha, \;\; |q^2(x)|_2 \le \alpha, \;\; \text{a.e. } x \in \Omega.$$

Continuous generalization of max-flow problem

Primal-dual formulation

$$\inf_{\phi^1,\phi^2} \sup_{p_s,p_t,p^{12},q} \int_\Omega \{(1-\phi^1)p_s^1 + (1-\phi^2)p_s^2\}(x)\,dx$$

$$+ \int_\Omega \phi^1(x)p_t^1(x) + \phi^2(x)p_t^2(x) + (\phi^1(x)-\phi^2(x))p^{12}(x)\,dx$$

$$+ \int_\Omega \phi^1(x)\,\mathrm{div}\,q^1(x)\,dx + \int_\Omega \phi^2(x)\,\mathrm{div}\,q^2(x)\,dx,$$

subject to

$$p_s^1(x) \le C(x), \quad p_s^2(x) \le D(x), \quad p_t^1(x) \le A(x), \quad p_t^2 \le B(x),$$
$$-F(x) \le p^{12}(x) \le E(x), \quad |q^1(x)|_2 \le \alpha, \quad |q^2(x)|_2 \le \alpha, \quad \text{a.e. } x \in \Omega.$$

Continuous generalization of max-flow problem

Primal problem

$$\min_{\phi^1,\phi^2 \in \{0,1\}} \ \alpha \int_\Omega |\nabla \phi^1| + \alpha \int_\Omega |\nabla \phi^2|$$

$$\int_\Omega (1-\phi^1(x))C(x) + (1-\phi^2(x))D(x) + \phi^1(x)A(x) + \phi^2(x)B(x)\,dx$$

$$+ \int_\Omega \max\{\phi^1(x)-\phi^2(x),0\}E(x) - \min\{\phi^1(x)-\phi^2(x),0\}F(x)\,dx$$

$$\text{subject to } \phi^1(x),\phi^2(x) \in [0,1] \text{ a.e. } x \in \Omega$$

Max-flow algorithm

Augmented Lagrangian Problem

$$\sup_{p_s, p_t, p^{12}, q} \inf_{\phi^1, \phi^2} L(p_s, p_t, p^{12}, q, \phi) = \int_\Omega p_s^1(x) + p_s^2(x)\, dx$$

$$+ \sum_{i=1}^{2} \int_\Omega \phi^i(x)(\operatorname{div} q^i(x) - p_s^i(x) + p_t^i(x) + (-1)^{i+1} p^{12}(x))\, dx$$

$$- \frac{c}{2} \sum_{i=1}^{2} || \operatorname{div} q^i(x) - p_s^i(x) + p_t^i(x) + (-1)^{i+1} p^{12}(x)||^2$$

$$p_s^1(x) \leq C(x), \quad p_s^2(x) \leq D(x),$$
$$p_t^1(x) \leq A(x), \quad p_t^2 \leq B(x),$$

$$-F(x) \leq p^{12}(x) \leq E(x),$$

$$|q^1(x)|_2 \leq \alpha, \quad |q^2(x)|_2 \leq \alpha, \quad \forall x \in \Omega.$$

Max-flow algorithm

Augmented Lagrangian Method (ADMM)

Initialize p_s^0, p_t^0, ${p^{12}}^0$, q^0, ϕ^0, for $k = 0, 1, \ldots$

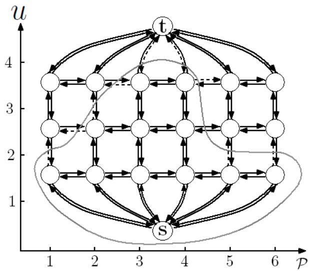
$$
\begin{aligned}
{p_s^i}^{k+1} &:= \arg \max_{p_s^i(x) \le C_s^i(x)} L_c(p_s^i, {p_t^i}^k, {q^i}^k, \phi^k), \;\; i = 1,2 \\
{p^{12}}^{k+1} &:= \arg \max_{-C^{21}(x) \le p^{12}(x) \le C^{12}(x)} L_c({p_s^i}^{k+1}, {p_t^i}^k, {q^i}^k, \phi^k), \;\; i = 1,2 \\
{q^i}^{k+1} &:= \arg \max_{|q^i| \le \alpha} L_c(p_s^{k+1}, p_t^k, q, \phi^k), \;\; i = 1,2 \\
{p_t^i}^{k+1} &:= \arg \max_{p_t^i(x) \le C_t^i(x)} L_c({p_s^i}^{k+1}, {p_t^i}^k, {p^{12}}^{k+1}, {q^i}^k, \phi^k), \;\; i = 1,2 \\
{\phi^i}^{k+1} &= {\phi^i}^k - c\,(\operatorname{div} {q^i}^{k+1}(-{p_s^i}^{k+1} + {p_t^i}^{k+1} + (-1)^{i+1} {p^{12}}^{k+1}), \;\; i = 1,2
\end{aligned}
$$

Max-flow algorithm

	Simple image			Brain image		
	iter	flops/iter	flops	iter	flops/iter	flops
Chan-Vese MF	11	$9.94 * 10^5$	$1.09 * 10^7$	60	$4.20 * 10^6$	$2.52 * 10^8$
Special MF	12	$3.92 * 10^6$	$4.70 * 10^7$	60	$1.65 * 10^7$	$9.93 * 10^8$
Potts MF	12	$5.07 * 10^6$	$6.09 * 10^7$	60	$2.14 * 10^7$	$1.29 * 10^9$
Simplex MF	60	$1.96 * 10^6$	$1.18 * 10^8$	190	$8.21 * 10^6$	$1.56 * 10^9$
Pock MF	295	$1.12 * 10^7$	$3.31 * 10^9$	1020	$4.71 * 10^7$	$4.79 * 10^{10}$

Table: Comparisons with other relaxations implemented with similar max-flow algorithm (MF). Number of iterations k, number of flops per iteration and total number of flops to reach energy precision $\frac{E^k - E^*}{E^*} < 10^{-3}$.

Labeling function representation



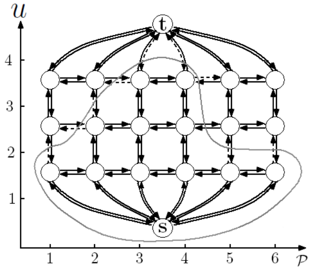
2	3	4	4	2	2

- H. Ishikawa 2003, Exact optimization for Markov random fields with convex priors, IEEE PAMI Volume 25 Issue 10, Page 1333-1336
- Pock et al. 2010/2008, Global Solutions of Variational Models with Convex Regularization, SIAM J. Imaging Sci., 3(4), 1122–1145, ECCV
- Bae et al.: A Fast Continuous Max-Flow Approach to Non-Convex Multi-Labeling Problems, LNCS 8293, pg. 134–154, 2014

Graph cut minimization labeling function

$$\min_{u\,:\,\Omega\mapsto\{1,\ldots,n\}} \sum_{p\in\mathcal{V}} \rho(u_p, p) + \alpha \sum_{(p,q)\in\mathcal{N}\subset\mathcal{V}\times\mathcal{V}} g^{\text{convex}}(|u_p - u_q|)\,.$$

Minimal cut on graph $\leftrightarrow$ minimizer of energy



1D illustration: Example of cut

2	3	4	4	2	2

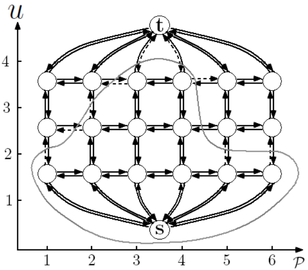
Corresponding labeling

H. Ishikawa 2003, Exact optimization for Markov random fields with convex priors, IEEE PAMI Volume 25 Issue 10, Page 1333-1336

Graph cut minimization labeling function

$$\min_{u\,:\,\Omega\mapsto\{1,\ldots,n\}} \sum_{p\in\mathcal{V}} \rho(u_p,p) + \alpha \sum_{(p,q)\in\mathcal{N}\subset\mathcal{V}\times\mathcal{V}} |u_p - u_q| \ .$$

Minimal cut on graph $\leftrightarrow$ minimizer of energy



1D illustration: Example of cut

2	3	4	4	2	2

Corresponding labeling

H. Ishikawa 2003, Exact optimization for Markov random fields with convex priors, IEEE PAMI Volume 25 Issue 10, Page 1333-1336

Convex relaxation labeling function

$$\min_{u\,:\,\Omega\mapsto\mathbb{R}} \int_\Omega \rho(u(x),x)\,dx + \int_\Omega g^{\text{convex}}(|\nabla u(x)|)\,dx\,,$$

Pock et al. proposed to represent u through binary function $\lambda\,:\Omega\times\mathbb{R}\mapsto\{0,1\}$

$$\lambda(x,\ell)\ :=\ \begin{cases} 1\,, & \text{if } u(x)>\ell \\ 0\,, & \text{if } u(x)\leq\ell \end{cases}\ .$$

Problem was expressed in terms of λ as

$$\min_{\lambda(\ell,x)\in\{0,1\}} \int_{\ell_{\min}}^{\ell_{\max}}\int_\Omega \left\{\alpha\,|\nabla_x\lambda| + \rho(\ell,x)\,|\partial_\ell\lambda(\ell,x)|\,\right\} dx d\ell\,.$$

subject to

$$\lambda(\ell_{\min},x)=1\,,\quad \lambda(\ell_{\max},x)=0\,,\quad x\in\Omega$$

Pock et al. 2010, Global Solutions of Variational Models with Convex Regularization, SIAM J. Imaging Sci., 3(4), 1122–1145

Convex relaxation labeling function

$$\min_{u:\,\Omega\mapsto\mathbb{R}} \int_\Omega \rho(u(x),x)\,dx + \alpha\int_\Omega |\nabla u(x)|\,dx\,,$$

Pock et al. proposed to represent u through binary function $\lambda\;:\Omega\times\mathbb{R}\mapsto\{0,1\}$

$$\lambda(x,\ell)\;:=\begin{cases}1\,, & \text{if } u(x)>\ell\\ 0\,, & \text{if } u(x)\leq\ell\end{cases}\;.$$

Problem was expressed in terms of λ as

$$\min_{\lambda(\ell,x)\in\{0,1\}}\int_{\ell_{\min}}^{\ell_{\max}}\int_\Omega\left\{\alpha\,|\nabla_x\lambda| + \rho(\ell,x)\,|\partial_\ell\lambda(\ell,x)|\,\right\}dxd\ell\,.$$

subject to

$$\lambda(\ell_{\min},x)=1\,,\quad \lambda(\ell_{\max},x)=0\,,\quad x\in\Omega$$

Pock et al. 2010, Global Solutions of Variational Models with Convex Regularization, SIAM J. Imaging Sci., 3(4), 1122–1145

Convex relaxation labeling function

$$\min_{u\,:\,\Omega\mapsto\mathbb{R}} \int_\Omega \rho(u(x),x)\,dx + \alpha\int_\Omega |\nabla u(x)|\,dx\,,$$

Pock et al. proposed to represent u through binary function $\lambda\,:\Omega\times\mathbb{R}\mapsto\{0,1\}$

$$\lambda(x,\ell)\;:=\begin{cases} 1\,, & \text{if } u(x)>\ell \\ 0\,, & \text{if } u(x)\leq\ell \end{cases}\;.$$

Problem was expressed in terms of λ as

$$\min_{\lambda(\ell,x)\in[0,1]} \int_{\ell_{\min}}^{\ell_{\max}}\int_\Omega \left\{\alpha\,|\nabla_x\lambda| + \rho(\ell,x)\,|\partial_\ell\lambda(\ell,x)|\,\right\} dxd\ell\,.$$

subject to

$$\lambda(\ell_{\min},x)=1\,,\quad \lambda(\ell_{\max},x)=0\,,\quad x\in\Omega$$

Pock et al. 2010, Global Solutions of Variational Models with Convex Regularization, SIAM J. Imaging Sci., 3(4), 1122–1145

Convex relaxation labeling function

Discrete labels

$$\min_{u\,:\,\Omega\mapsto\{1,\ldots,n\}} \int_\Omega \rho(u(x),x)\,dx + \alpha \int_\Omega |\nabla u(x)|\,dx\,,$$

Can be written as

$$\min_{\{\lambda_i\}_{i=1}^{n-1}\,:\,\Omega\mapsto\{0,1\}} \sum_{i=1}^{n} \int_\Omega (\lambda_{i-1}-\lambda_i)\,\rho(\ell_i,x)\,dx \;+\; \alpha \sum_{i=1}^{n-1} \int_\Omega |\nabla \lambda_i|\,dx$$

$$1 = \lambda_0(x) \geq \lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_{n-1}(x) \geq \lambda_n(x) = 0\ \forall\, x \in \Omega.$$

u related to λ by

$$u \;=\; \sum_{i=1}^{n} (\lambda_{i-1}-\lambda_i)\ell_i$$

Hochbaum 02, Darbon 06, Chambolle 05, Pock 09

Convex relaxation labeling function

Discrete labels

$$\min_{u\,:\,\Omega\mapsto\{1,\ldots,n\}} \int_\Omega \rho(u(x),x)\,dx + \alpha\int_\Omega |\nabla u(x)|\,dx\,,$$

Convex relaxation

$$\min_{\{\lambda_i\}_{i=1}^{n-1}\,:\,\Omega\mapsto[0,1]} \sum_{i=1}^{n}\int_\Omega (\lambda_{i-1}-\lambda_i)\,\rho(\ell_i,x)\,dx \;+\;\alpha\sum_{i=1}^{n-1}\int_\Omega |\nabla\lambda_i|\,dx$$

$$1=\lambda_0(x)\geq\lambda_1(x)\geq\lambda_2(x)\geq\ldots\geq\lambda_{n-1}(x)\geq\lambda_n(x)=0\;\forall\;x\in\Omega.$$

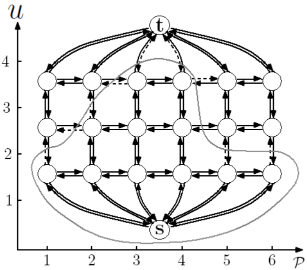
u related to λ by

$$u\;=\;\sum_{i=1}^{n}(\lambda_{i-1}-\lambda_i)\ell_i$$

Graph cut minimization labeling function (recall)

$$\min_{u:\,\Omega\mapsto\{1,\ldots,n\}} \sum_{p\in\mathcal{V}} \rho(u_p,p) + \alpha \sum_{(p,q)\in\mathcal{N}\subset\mathcal{V}\times\mathcal{V}} |u_p - u_q| \;.$$

Minimal cut on graph $\leftrightarrow$ minimizer of energy



1D illustration: Example of cut

2	3	4	4	2	2

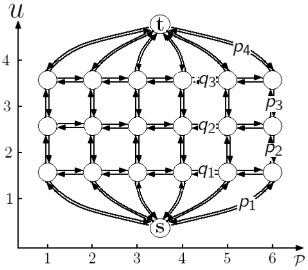
Corresponding labeling

Corresponding discrete max-flow problem

$$\sup_{p,q} \int_\Omega p_1(x)\,dx$$

$$\begin{aligned}
&|q_i(x)|_1 = |q_i^1| + |q_i^2| \le \alpha && \text{for } x \in \Omega\,,\ \ i = 1,\dots,n-1\\
&p_i(x) \le \rho(\ell_i, x) && \text{for } x \in \Omega\,,\ \ i = 1,\dots,n\\
&\big(\operatorname{div} q_i - p_i + p_{i+1}\big)(x) = 0 && \text{for } x \in \Omega\,,\ \ i = 1,\dots,n-1\\
&q_i \cdot n = 0 && \text{on } \partial\Omega\,,\ \ i = 1,\dots,n-1\,.
\end{aligned}$$

- 1D illustration



Continuous max-flow problem

$$\sup_{p,q} \int_\Omega p_1(x)\,dx$$

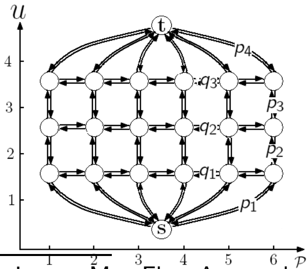
$$|q_i(x)|_2 = \sqrt{\left|q_i^1\right|^2 + \left|q_i^2\right|^2} \leq \alpha \quad \text{for } x \in \Omega\,, \quad i = 1, \ldots, n-1$$

$$p_i(x) \leq \rho(\ell_i, x) \quad \text{for } x \in \Omega\,, \quad i = 1, \ldots, n$$

$$\left(\operatorname{div} q_i - p_i + p_{i+1}\right)(x) = 0 \quad \text{for } x \in \Omega\,, \quad i = 1, \ldots, n-1$$

$$q_i \cdot n = 0 \quad \text{on } \partial\Omega\,, \quad i = 1, \ldots, n-1\,.$$

- 1D illustration



Bae et al.: A Fast Continuous Max-Flow Approach to Non-Convex Multi-Labeling Problems, LNCS 8293, pg. 134-154, 2014

Continuous max-flow problem

$$\sup_{p,q} \int_\Omega p_1(x)\, dx$$

$$\begin{aligned}
&|q_i(x)|_2 = \sqrt{\left|q_i^1\right|^2 + \left|q_i^2\right|^2} \leq \alpha && \text{for } x \in \Omega\,,\;\; i = 1, \ldots, n-1 \\
&p_i(x) \;\leq\; && \rho(\ell_i, x) \text{for } x \in \Omega\,,\;\; i = 1, \ldots, n \\
&\big(\operatorname{div} q_i - p_i + p_{i+1}\big)(x) \;=\; 0 && \text{for } x \in \Omega\,,\;\; i = 1, \ldots, n-1 \\
&q_i \cdot n = 0 && \text{on } \partial\Omega\,,\;\; i = 1, \ldots, n-1\,.
\end{aligned}$$

Lagrange multipliers λ_i for flow conservation constraints results in

$$\inf_\lambda \sup_{p,q} \int_\Omega \Big\{ p_1 + \sum_{i=1}^{n-1} \lambda_i \big(\operatorname{div} q_i - p_i + p_{i+1} \big) \Big\}\, dx$$

$$p_i(x) \;\leq\; \rho(\ell_i, x)\,, \quad |q_i(x)|_2 \leq \alpha \;\; x \in \Omega$$

Continuous max-flow problem

$$\sup_{p,q} \int_\Omega p_1(x)\, dx$$

$$\begin{aligned}
&|q_i(x)|_2 = \sqrt{\left|q_i^1\right|^2 + \left|q_i^2\right|^2} \le \alpha && \text{for } x \in \Omega\,, \;\; i = 1, \dots, n-1 \\
&p_i(x) \le \rho(\ell_i, x) && \text{for } x \in \Omega\,, \;\; i = 1, \dots, n \\
&\big(\operatorname{div} q_i - p_i + p_{i+1}\big)(x) = 0 && \text{for } x \in \Omega\,, \;\; i = 1, \dots, n-1 \\
&q_i \cdot n = 0 && \text{on } \partial\Omega\,, \;\; i = 1, \dots, n-1\,.
\end{aligned}$$

Rearranged primal-dual formulation

$$\inf_\lambda \sup_{p,q} \sum_{i=1}^{n} \int_\Omega (\lambda_{i-1} - \lambda_i) p_i\, dx + \sum_{i=1}^{n-1} \int_\Omega \lambda_i \operatorname{div} q_i\, dx$$

$$p_i(x) \le \rho(\ell_i, x)\,, \quad |q_i(x)|_2 \le \alpha \;\; x \in \Omega$$

Continuous max-flow problem

Rearranged primal-dual formulation (repeat)

$$\inf_{\lambda} \sup_{p,q} \sum_{i=1}^{n} \int_{\Omega} (\lambda_{i-1} - \lambda_i) p_i \, dx \; + \; \sum_{i=1}^{n-1} \int_{\Omega} \lambda_i \operatorname{div} q_i \, dx$$

$$p_i(x) \;\leq\; \rho(\ell_i, x)\,, \quad |q_i(x)|_2 \leq \alpha \;\; x \in \Omega$$

Leads back to primal formulation

$$\min_{\{\lambda_i\}_{i=1}^{n-1} : \Omega \mapsto \{0,1\}} \; \sum_{i=1}^{n} \int_{\Omega} (\lambda_{i-1} - \lambda_i)\, \rho(\ell_i, x)\, dx \; + \; \alpha \sum_{i=1}^{n-1} \int_{\Omega} |\nabla \lambda_i| \, dx$$

$$1 = \lambda_0(x) \geq \lambda_1(x) \geq \lambda_2(x) \geq \ldots \geq \lambda_{n-1}(x) \geq \lambda_n(x) = 0 \;\forall\; x \in \Omega.$$

Augmented Lagrangian algorithm

$$L_c(p,q,\lambda) \;:=\; \int_\Omega p_1 + \sum_{i=1}^{n-1} \lambda_i(\operatorname{div} p_i + p_{i+1} - p_i) - \frac{c}{2}|\operatorname{div} p_i + p_{i+1} - p_i|^2 \, dx,$$

Init. p^1, q^1 and λ^1, let $k, i = 1$. For $k = 1, \ldots$

- For each layer $i = 1 \ldots n$ solve

$$q_i^{k+1} \;:=\; \arg \max_{\|q\|_\infty \le \alpha} \; L_c((\tilde{p}_{i\le j}^{k+1}, p_{i>j}^k), (q_{j<i}^{k+1}, q_i, q_{j>i}^k), \lambda^k)$$

$$p_i^{k+1} \;:=\; \arg \max_{p_i(x) \le \rho(\ell_i, x)\, \forall x \in \Omega} \; L_c((p_{j<i}^{k+1}, p_i, p_{j>i}^k), (q_{j\le i}^{k+1}, q_{j>i}^k), \lambda^k)\,,$$

- Update multipliers λ_i, $i = 1, \ldots, n-1$, by

$$\lambda_i^{k+1} \;=\; \lambda_i^k - c\,(\operatorname{div} q_i^{k+1} - p_i^{k+1} + p_{i+1}^{k+1})\,;$$

Stereo reconstruction application

Given two images I_L and I_R taken from slightly different viewpoints

Want to reconstruct "depth" u by minimizing

$$\min_{u\,:\,\Omega \mapsto \{1,\ldots,16\}} \int_\Omega \rho(u(x),x)\,dx + \alpha \int_\Omega |\nabla u| \;,$$

$$\rho(u,x) = \sum_{j=1}^{3} |I_L^j(x) - I_R^j(x + (u,0)^T)|.$$

Stereo reconstruction application

graph cut 4 neighbors

graph cut 8 neighbors

Pock et al.

Continuous max-flow

Stereo reconstruction application

$\varepsilon < 10^{-4}$		$\varepsilon < 10^{-5}$		$\varepsilon < 10^{-6}$	
Primal-dual	Max-flow	Primal-dual	Max-flow	Primal-dual	Max-flow
14305	920 ($\times$ 5)	$>$ 30000	1310 ($\times$ 5)	$>$ 30000	1635 ($\times$ 5)

Table: Iteration counts for stereo experiment. Number of iterations to reach an energy precision of 10^{-4}, 10^{-5} and 10^{-6}.

$\varepsilon = \frac{E^i - E^*}{E^*}$, E^i is energy at iteration i and E^* is energy of final solution.

Convex relaxations for other related NP-hard problems

Will derive convex relaxations for

- Total curve length (Potts' regularization term)
- ℓ_0 of gradient
- joint minimization over regions and region parameters in segmentation
- non-submodular data terms

Convex relaxations for total curvelength (Potts' regularizer)

- Chambolle et al. 2012/2009, A Convex Approach to Minimal Partitions, SIAM J. Imaging Sci., 5(4), 1113–1158.
- Bae and Tai, Efficient Global Minimization Methods for Image Segmentation Models with Four Regions, Journal of Mathematical Imaging and Vision, 2014
- Simplex constrained relaxation (Zach et al. 08, Lellmann et al. 09, Bae et al. 09/11) was covered in previous talk

Convex relaxations for total curvelength (Potts' regularizer)

$$\min_{\{\lambda_i\}_{i=1}^{n-1}} \sup_{\{q_i\}_{i=1}^{n-1}} \sum_{i=1}^{n} \int_\Omega (\lambda_{i-1} - \lambda_i)\, \rho(\ell_i, x)\, dx \,+\, \alpha \sum_{i=1}^{n-1} \int_\Omega \lambda_i \operatorname{div} q_i \, dx$$

s.t. $1 = \lambda_0(x) \geq \lambda_1(x) \geq \lambda_2(x) \geq ... \geq \lambda_{n-1}(x) \geq \lambda_n(x) = 0 \;\; x \in \Omega.$

$|q_i|_\infty \leq 1\,, \;\; i = 1, ..., n$

Pock et al 09,12 proposed to avoid multiple countings of boundaries by optimize over the convex set

$$q(x) \in \Big\{ q \in \mathbb{R}^{n \times N} \; | \quad |\sum_{i=i_1}^{i_2} q_i| \leq \alpha \; ;$$
$$\forall \, (i_1, i_2) \,, \quad 1 \leq i_1 \leq i_2 \leq n-1 \; \Big\} , \;\; x \in \Omega$$

and relax binary constraint by $\lambda_i(x) \in [0, 1]$, $x \in \Omega\,,\; i = 1, ..., n$

- Advantage: tighest relaxation for Potts' model
- Disadvantage: number of contraints grow quadratically in number of regions

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Chambolle et al. 2012/2009, A Convex Approach to Minimal Partitions, SIAM J. Imaging Sci., 5(4), 1113–1158.

Convex relaxations for total curvelength (Potts' regularizer)

Potts model in terms of overlapping binary functions

$$\min_{\phi^1,\phi^2} \sup_{|q^1|,|q^2|\leq 1} \alpha \int_\Omega \phi^1 \operatorname{div} q^1 dx + \alpha \int_\Omega \phi^2 \operatorname{div} q^2 \, dx,$$

$$+ \int_\Omega \phi^1(x) C_t^1(x) + \phi^2(x) C_t^2(x) \, dx$$

$$+ \int_\Omega \max\{\phi^1(x) - \phi^2(x), 0\} C^{12}(x) \, dx - \int_\Omega \min\{\phi^1(x) - \phi^2(x), 0\} C^{21}(x) \, dx$$

$$+ \int_\Omega (1 - \phi^1(x)) C_s^1(x) + (1 - \phi^2(x)) C_s^2(x) \, dx$$

$$\text{such that } \phi^1(x), \phi^2(x) \in [0,1] \ \forall x \in \Omega$$

- Convex relaxation of Potts model by adding additional dual constraints

$$|q^1(x) + q^2(x)| \leq 1, \quad |q^1(x) - q^2(x)| \leq 1, \quad \forall x \in \Omega$$

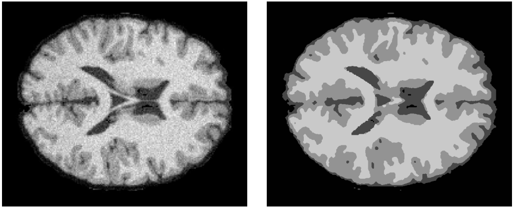
Bae and Tai, Efficient Global Minimization Methods for Image Segmentation Models with Four Regions, Journal of Mathematical Imaging and Vision, 2014

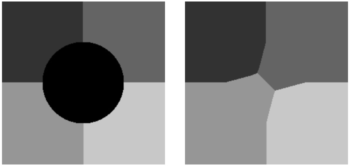
Convex relaxations for total curvelength (Potts' regularizer)

Example of other regularization term

Constraint set that disfavors interphases between region 1 and region 4

$$|q^1(x)| \leq \alpha, \quad |q^2(x)| \leq \alpha, \quad |q^1(x) + q^2(x)| \leq 1, \quad \forall x \in \Omega$$





Convex relaxations for total curvelength (Potts' regularizer)

Different representations of partitions in terms of functions

- 1) Vector function: $\mathbf{u}(x) = (u^1(x), ..., u^n(x)) = e_i$ for $x \in \Omega_i$
- 2) Labeling function: $\ell(x) = i$ for all $x \in \Omega_i$
- 3) Intersection of $m = \log_2(n)$ binary functions $\phi^1, ...\phi^m$

$x \in \Omega_1$ iff	$\mathbf{u}(x) = e_1$	$\ell(x) = 1$	$\phi^1(x) = 1, \phi^2(x) = 0$
$x \in \Omega_2$ iff	$\mathbf{u}(x) = e_2$	$\ell(x) = 2$	$\phi^1(x) = 1, \phi^2(x) = 1$
$x \in \Omega_3$ iff	$\mathbf{u}(x) = e_3$	$\ell(x) = 3$	$\phi^1(x) = 0, \phi^2(x) = 0$
$x \in \Omega_4$ iff	$\mathbf{u}(x) = e_4$	$\ell(x) = 4$	$\phi^1(x) = 0, \phi^2(x) = 1$

Table: Representation of 4 regions.

Convex relaxations for total curvelength (Potts' regularizer)

- Relaxation over unit simplex (Zach et al. 08, Lellmann et al. 09, Bae et al. 09/11)

$$u_i(x) \;=\; I_{\Omega_i}(x) \;:=\; \begin{cases} 1, & x \in \Omega_i \\ 0, & x \notin \Omega_i \end{cases} \;, \qquad i = 1, \ldots, n$$

- Problem can be expressed as

$$\min_{u_i(x) \in \{0,1\}} \sum_{i=1}^{n} \int_\Omega u_i(x) f_i(x)\, dx \;+\; \alpha \sum_{i=1}^{n} \int_\Omega |\nabla u_i| \; dx \,,$$

$$\text{s.t.} \;\; \sum_{i=1}^{n} u_i(x) \;=\; 1 \;\;, u_i(x) \in \{0,1\} \;\; \forall x \in \Omega, \;\; i = 1, ..., n$$

- Convex relaxation: $u_i(x) \in [0,1] \; \forall x \in \Omega, \; i = 1, ..., n$
  - If computed solution is binary: also global minimum
  - else: convert into binary by rounding schemes (not generally exact)

Image reconstruction with sparse gradients

- Piecewise constant Mumford-Shah model

$$\inf_{\Gamma, I \in X} \lim_{\lambda \to \infty} \int_{\Omega} |I(x) - I^0(x)|^\beta dx + \lambda \int_{\Omega \setminus \Gamma} |\nabla I|^2 dx + \alpha \int_{\Gamma} ds.$$

- Can also be expressed as a partition problem where the number of regions n is unknown.

$$\min_{n} \min_{\{\Omega_i\}_{i=1}^n} \min_{\{\mu_i\}_{i=1}^n \in X} \quad \sum_{i=1}^{n} \int_{\Omega_i} |\mu_i - I^0(x)|^\beta dx \ + \ \alpha \sum_{i=1}^{n} \int_{\partial \Omega_i} ds$$

where $I(x) = \mu_i$ for all $x \in \Omega_i$, $i = 1, ..., n$.

- Discrete version can be formulated as

$$\min_{I \in X} ||I - I^0||_2^\beta + \alpha ||\nabla I||_0$$

- The set of gray values X can be continuous $X = [0, 1]$, or discretized $X = \{\ell_1, ..., \ell_L\}$ e.g. $X = \{0, 1/L, 2/L, ..., 1\}$

Image reconstruction with sparse gradients

- PC Mumford-Shah model with $X = \{\ell_1, ..., \ell_L\}$

$$\inf_{\Gamma, I \in X} \lim_{\lambda \to \infty} \int_\Omega |I(x) - I^0(x)|^\beta dx + \lambda \int_{\Omega \setminus \Gamma} |\nabla I|^2 dx + \alpha \int_\Gamma ds.$$

- Reformulated problem

$$\min_u \sum_{i=1}^{L} \int_\Omega u_i(x)|I^0(x) - \ell_i|^\beta + \alpha |\nabla u_i| \, dx$$

subject to

$$\sum_{i=1}^{L} u_i(x) = 1 \,, \quad \forall x \in \Omega$$

$$u_i(x) \in \{0, 1\} \,, \quad \forall x \in \Omega, \ \ i = 1, ..., L$$

Theorem: Given a minimizer u^*. Define $I = \sum_{i=1}^{L} \ell_i u_i^*$, then I is a global minimizer to the quantized piecewise constant Mumford-Shah model with $X = \{\ell_1, ..., \ell_L\}$.

Image reconstruction with sparse gradients

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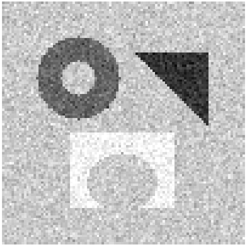
Image reconstruction with sparse gradients



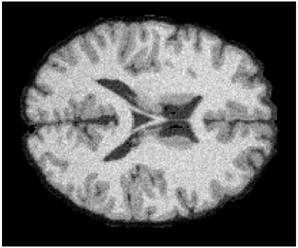
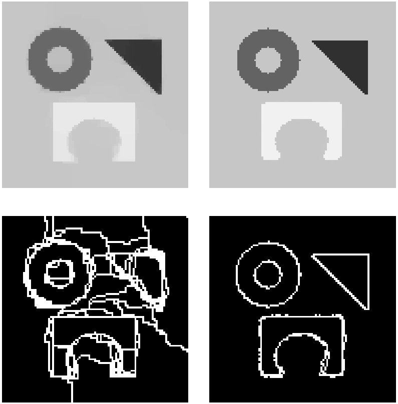
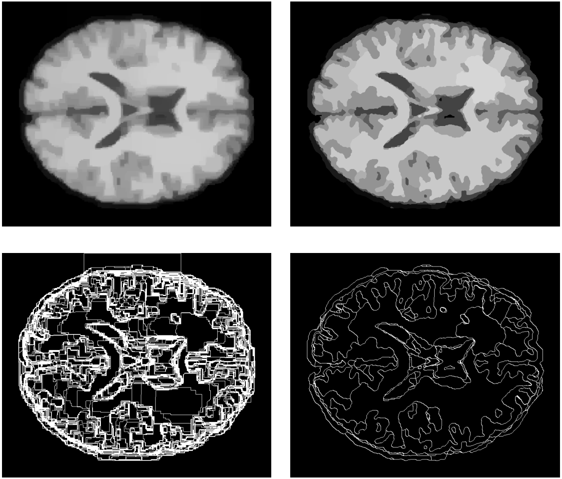


Image reconstruction with sparse gradients



Left: ℓ_1 relaxation (total variation), Right: New convex relaxation. Bottom: Set of pixels with non-zero gradient ∇I. Set of gray values quantized to $X = \{0, 1, ..., 255\}$.

Image reconstruction with sparse gradients



Left: total variation (ℓ_1 relaxation), Right: New relaxation. Bottom: Set of pixels with non-zero gradient ∇I. Set of gray values quantized to $X = \{0, 1, ..., 255\}$.

Convex relaxation for parametric image segmentation

- We are interested in problem

$$\min_{\{\Omega_i\}_{i=1}^n,\{\mu_i\}_{i=1}^n \in X} \sum_{i=1}^n \int_{\Omega_i} |I^0(x) - \mu_i|^\beta \, dx + \alpha \sum_{i=1}^n \int_{\partial\Omega_i} ds,$$

$$\text{such that} \quad \cup_{i=1}^n \Omega_i = \Omega, \quad \cap_{i=1}^n \Omega_i = \emptyset$$

- Equivalent formulation

$$\min_{u_i(x)\in\{0,1\}} \min_{\mu_i \in X} \quad \sum_{i=1}^n \int_\Omega u_i(x) |I^0(x) - \mu_i|^\beta \, dx + \alpha \sum_{i=1}^n \int_\Omega |\nabla u_i| \, dx\,.$$

$$\text{such that} \quad \sum_{i=1}^n u_i(x) = 1, \quad \forall x \in \Omega$$

- We assume X is discrete $X = \{\ell_1, ..., \ell_L\}$
- For instance $X = \{1, ..., 255\}$ or $X = \{0, 1/L, 2/L, ..., 1\}$

Convex relaxation for parametric image segmentation

Related work with parameters:

- Alternating minimization w.r.t. parameters and regions: no guarantee of global minimizers
- Darbon 07, Lempitsky 08, Strandmark 09 considered $n = 2$ and avoided checking all L^2 combinations
- Brown et al. 2011: optimization of binary function defined over space of $O(L^2 \cdot |\Omega|)$ dimensions for $n = 2$

Our work:

- Size of convex problem grows linearly in L, i.e. as $O(L \cdot |\Omega|)$, and can handle any number of regions n, known or unknown.

Convex relaxation for parametric image segmentation

Reformulation as minimal binary function over $\Omega \times X$

- For each discrete gray value ℓ_i define a binary function $u_i \,:\, \Omega \mapsto \{0,1\},\ i = 1,...,L$
- Define minimization problem

$$\min_u \sum_{i=1}^{L} \int_\Omega u_i(x)|I^0(x) - \ell_i|^\beta + \alpha\,|\nabla u_i|\,dx$$

subject to

$$\sum_{i=1}^{L} u_i(x) = 1\,, \quad \forall x \in \Omega$$

$$\sum_{i=1}^{L} \sup_{x\in\Omega} u_i(x) \leq n$$

$$u_i(x) \in \{0,1\}\,, \quad \forall x \in \Omega, \quad i = 1,...,L$$

Convex relaxation for parametric image segmentation

$$\min_u \sum_{i=1}^{L} \int_\Omega u_i(x)|I^0(x) - \ell_i|^\beta + \alpha\,|\nabla u_i|\,dx$$

subject to

$$\sum_{i=1}^{L} u_i(x) = 1\,, \quad \forall x \in \Omega$$

$$\sum_{i=1}^{L} \sup_{x\in\Omega} u_i(x) \le n$$

$$u_i(x) \in \{0,1\}\,, \quad \forall x \in \Omega, \;\; i = 1, ..., L$$

Theorem: Given an optimizer u^*. Let n^* be the number of indices i for which $u_i^* \not\equiv 0$. Define the set of indices $\{i_j\}_{j=1}^{n^*} \subset \{1, ..., L\}$ such that $u_{i_j}^* \not\equiv 0$. Then $\{u_{i_j}^*\}_{j=1}^{n^*}, \{\ell_{i_j}\}_{j=1}^{n^*}$ is a global optimizer to the original problem. $I = \sum_{i=1}^{L} \ell_i u_i$ is an optimal piecewise constant function.

Convex relaxation for parametric image segmentation

$$\min_u \sum_{i=1}^{L} \int_\Omega u_i(x)|I^0(x) - \ell_i|^\beta + \alpha\,|\nabla u_i|\,dx$$

subject to

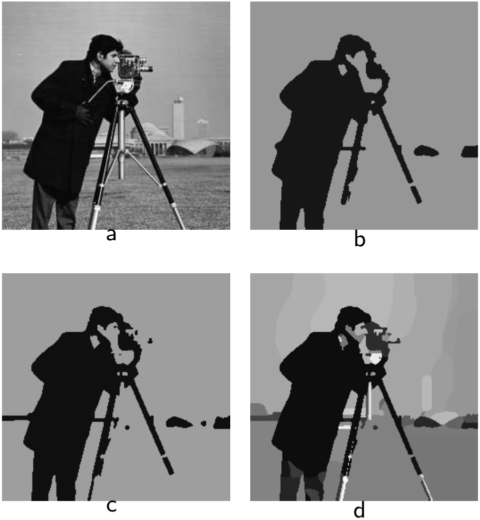
$$\sum_{i=1}^{L} u_i(x) = 1\,, \quad \forall x \in \Omega$$

$$\sum_{i=1}^{L} \sup_{x\in\Omega} u_i(x) \le n$$

$$u_i(x) \in [0,1]\,, \quad \forall x \in \Omega, \quad i = 1, ..., L$$

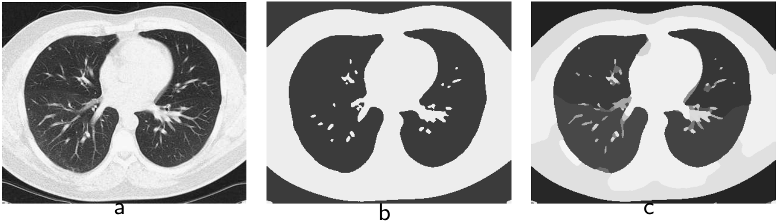
Theorem: Given an optimizer u^*. Let n^* be the number of indices i for which $u_i^* \not\equiv 0$. Define the set of indices $\{i_j\}_{j=1}^{n^*} \subset \{1, ..., L\}$ such that $u_{i_j}^* \not\equiv 0$. Then $\{u_{i_j}^*\}_{j=1}^{n^*}, \{\ell_{i_j}\}_{j=1}^{n^*}$ is a global optimizer to the original problem. $I = \sum_{i=1}^{L} \ell_i u_i$ is an optimal piecewise constant function.

Convex relaxation for parametric image segmentation



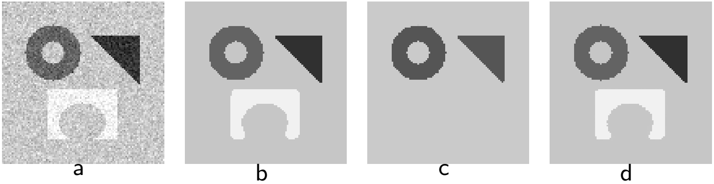
(a) Input image. (b) convex relaxation L_1 data term $|I^0(x) - \mu_i|$, (c) convex relaxation L^2 data term $|I^0(x) - \mu_i|^2$, (d) piecewise constant Mumford-Shah model.

Convex relaxation for parametric image segmentation



(a) Input image. (b) convex relaxation L_2 data term $|I^0(x) - \mu_i|^2$, (c) piecewise constant Mumford-Shah model.

Convex relaxation for parametric image segmentation



(a) Input. (b)-(c) Convex relaxation with: (b) $n = 4$, (c) $n = 2$.
(d) Convex relaxation of piecewise constant Mumford-Shah model.

Max-flow based algorithm

$$\min_u \sum_{i=1}^{L} \int_\Omega u_i(x)|I^0(x) - \ell_i|^\beta + \alpha |\nabla u_i| \, dx$$

subject to

$$\sum_{i=1}^{L} u_i(x) = 1\,, \quad \forall x \in \Omega$$

$$\sum_{i=1}^{L} \sup_{x\in\Omega} u_i(x) \leq n$$

$$u_i(x) \in [0,1]\,, \quad \forall x \in \Omega, \;\; i = 1, \ldots, L$$

Max-flow based algorithm

- Let γ be lagrange multiplier for the constraint

$$\sum_{i=1}^{L} \max_{x \in \Omega} u_i(x) - n = 0.$$

- Lagrangian formulation

$$\max_{\gamma} \min_{u} \mathcal{L}(u, \gamma)$$

$$= \sum_{i=1}^{L} \int_{\Omega} u_i(x) |I^0(x) - \ell_i|^{\beta} + \alpha \, |\nabla u_i| \, dx + \gamma \Big(\sum_{i=1}^{L} \max_{x \in \Omega} u_i(x) - n \Big)$$

$$\text{s.t.} \quad \sum_{i=1}^{L} u_i(x) = 1, \quad u_i(x) \geq 0 \; \forall x \in \Omega, \; i = 1, ..., L, \quad \gamma \geq 0$$

Max-flow based algorithm

$$\max_{\gamma} \min_{u} \mathcal{L}(u, \gamma)$$

$$= \sum_{i=1}^{L} \int_{\Omega} u_i(x) |I^0(x) - \ell_i|^{\beta} + \alpha \, |\nabla u_i| \, dx + \gamma(\sum_{i=1}^{L} \max_{x \in \Omega} u_i(x) - n)$$

$$\text{s.t. } \{\sum_{i=1}^{L} u_i(x) = 1, \quad u_i(x) \geq 0, \; i = 1, ..., L\} = \Delta_+ \; \forall x \in \Omega, \quad \gamma \geq 0$$

1. $u^{k+1} = \arg\min_{u} \mathcal{L}(u, \gamma^k), \quad \text{s.t. } u(x) \in \Delta_+ \; \forall \; x \in \Omega$

2. $\gamma^{k+1} = \max(0, \gamma^k + c(\sum_{i=1}^{L} \max_{x \in \Omega} u_i^{k+1}(x) - n))$

Max-flow based algorithm

$$\max_{\gamma} \min_{u} \mathcal{L}(u, \gamma)$$

$$= \sum_{i=1}^{L} \int_{\Omega} u_i(x) |I^0(x) - \ell_i|^{\beta} + \alpha \, |\nabla u_i| \, dx + \gamma(\sum_{i=1}^{L} \max_{x \in \Omega} u_i(x) - n)$$

$$\text{s.t. } \{\sum_{i=1}^{L} u_i(x) = 1, \quad u_i(x) \geq 0, \; i = 1, ..., L\} = \Delta_+ \; \forall x \in \Omega, \quad \gamma \geq 0$$

$$1. \; u^{k+1} = \arg\min_{u} \mathcal{L}(u, \gamma^k), \quad \text{s.t. } u(x) \in \Delta_+ \; \forall \; x \in \Omega$$

$$2. \; \gamma^{k+1} = \max(0, \gamma^k + c(\sum_{i=1}^{L} \max_{x \in \Omega} u_i^{k+1}(x) - n))$$

- 1. has form of image segmentation model with label cost prior (Zhu and Yuille 96)

$$\min_{n} \min_{\{\Omega_i\}_{i=1}^{n}} \sum_{i=1}^{n} \int_{\Omega_i} f_i(I^0(x)) \, dx + \sum_{i=1}^{n} \alpha \int_{\partial \Omega_i} ds + \gamma \cdot n,$$

Max-flow based algorithm

$$\min_u \sum_{i=1}^{L} \int_\Omega u_i(x)|I^0(x) - \ell_i|^\beta \, dx + \alpha \, |\nabla u_i| \, dx + \gamma \sum_{i=1}^{n} \max_{x\in\Omega} u_i(x)$$

$$\text{such that } \sum_{i=1}^{L} u_i(x) \; = \; 1, \;\; u_i(x) \in [0,1], \;\; \forall x \in \Omega, \;\; i = 1, ..., L$$

Primal-dual formulation in case $\gamma > 0$

$$\sup_{p_s, p, q} \; \inf_u \; \int_\Omega p_s \, dx \; + \; \sum_{i=1}^{L} \int_\Omega u_i (\operatorname{div} q_i - p_s + p_i - r_i) \, dx \Big\}$$

$$p_i(x) \; \leq \; f_i(x) \, , \quad |q_i(x)| \; \leq \; \alpha, \quad \int_\Omega |r_i(x)| \, dx \leq \gamma \, ; \quad i = 1 \dots L$$

Augmented Lagrangian functional

$$\int_\Omega p_s \, dx + \sum_{i=1}^{n} \langle u_i, \operatorname{div} q_i - p_s + p_i - r_i \rangle - \frac{c}{2} \sum_{i=1}^{n} \|\operatorname{div} q_i - p_s + p_i - r_i\|^2$$

Max-flow based algorithm

Augmented Lagrangian Method (ADMM):
Initialize p_s^0, p_i^0, q^0, r^0 and ϕ^0. For $k = 0, 1, \ldots$

$$
\begin{aligned}
q_i^{k+1} &:= \arg \max_{\|q_i\|_\infty \le \alpha} -\frac{c}{2} \left\| \operatorname{div} q_i + p_i^k(x) - p_s^k(x) - r_i^k(x) - u_i^k(x)/c \right\|^2 \\
p_i^{k+1} &:= \arg \max_{p_i(x) \le \rho(\ell_i, x)} \\
&\quad - \frac{c}{2} \left\| p_i + \operatorname{div} q_i^{k+1}(x) - p_s^k(x) - r_i^k(x) - u_i^k(x)/c \right\|^2 \\
r_i^{k+1} &:= \arg \max_{r_i(x) \in R_i^\gamma} -\frac{c}{2} \left\| r_i - \operatorname{div} q_i^{k+1}(x) + p_s^k(x) - p_i^k(x) + u_i^k(x)/c \right\|^2 \\
p_s^{k+1} &:= \arg \max_{p_s} \int_\Omega p_s \, dx \\
&\quad - \frac{c}{2} \sum_{i=1}^{n} \left\| p_s - p_i^{k+1}(x) - \operatorname{div} q_i^{k+1}(x) + r_i^{k+1}(x) + u_i^k(x)/c \right\|^2 \\
u_i^{k+1} &= u_i^k - c \left(\operatorname{div} q_i^{k+1} - p_s^{k+1} + p_i^{k+1} - r_i^{k+1} \right)
\end{aligned}
$$

Convex relaxation non-submodular data term

- Convex relaxation if $E(x)$ or $F(x)$ are negative for some $x \in \Omega$

$$\min_{\phi^1,\phi^2} \alpha \int_\Omega |\nabla \phi^1| + \alpha \int_\Omega |\nabla \phi^2|$$

$$\int_\Omega (1-\phi^1(x))C(x)+(1-\phi^2(x))D(x)+\phi^1(x)A(x)+\phi^2(x)B(x)\,dx$$

$$+\int_\Omega \{\max\{\phi^1-\phi^2,0\}\max(E,0)-\min\{\phi^1-\phi^2,0\}\max(F,0)\}(x)\,dx$$

$$\begin{cases} A(x)+B(x) & = f_2(x) \\ C(x)+D(x) & = f_3(x) \\ A(x)+E(x)+D(x) & = f_1(x) \\ B(x)+F(x)+C(x) & = f_4(x) \end{cases}$$

- Minimize over convex constraint $\phi^1, \phi^2 \in [0,1]$.
- **Theorem**: Binary solution obtained by thresholding ϕ^1, ϕ^2 at any level $t \in (0,1]$ is a **global** minimizer over $\phi^1, \phi^2 \in \{0,1\}$ under conditions which can be checked after computation.

Convex relaxation non-submodular data term

$$\inf_{\phi^1,\phi^2} \sup_{p_s,p_t,p^{12},q} \int_\Omega (1-\phi^1(x))p_s^1(x) + (1-\phi^2(x))p_s^2(x)\,dx$$

$$+ \int_\Omega \phi^1(x)p_t^1(x) + \phi^2(x)p_t^2(x)\,dx + \int_\Omega (\phi^1(x)-\phi^2(x))p^{12}(x)\,dx$$

$$+ \int_\Omega \phi^1(x)\,\mathrm{div}\, q^1(x)\,dx + \int_\Omega \phi^2(x)\,\mathrm{div}\, q^2(x)\,dx.$$

$$p_s^1(x) \le C(x), \quad p_s^2(x) \le D(x), \quad p_t^1(x) \le A(x), \quad p_t^2 \le B(x),$$
$$-F(x) \le p^{12}(x) \le E(x), |q^1(x)|_2, \quad \le \alpha, \quad |q^2(x)|_2 \le \alpha, \; \forall x \in \Omega.$$

- Minimize over convex constraint $\phi^1, \phi^2 \in [0,1]$.
- **Theorem**: Binary solution obtained by thresholding ϕ^1, ϕ^2 at any level $t \in (0,1]$ is a **global** minimum over $\phi^1, \phi^2 \in \{0,1\}$ provided

$$(A - p_t^1)(x) + (D - p_s^2)(x) \ge -E(x), \quad \forall x \in \Omega$$

$$(B - p_t^2)(x) + (C - p_s^1)(x) \ge -F(x), \quad \forall x \in \Omega,$$

Convex relaxation non-submodular data term

Summary

Exact minimization

- Focused on two approaches for multiphase problems with global optimality guarantee.
- Both could be formulated as max-flow/min-cut problems on a graph in discrete setting.
- Both could be exactly formulated as convex problems in continuous setting. Dual problems could be formulated as continuous max-flow problems.

Approximate minimization

- Presented convex relaxations for broader set of non-convex problems.
- Included Potts' model and joint optimization of regions and region parameters.
- Dual problems were formulated as max-flow, but now there may be a duality gap to original problems

References

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