Abstract

The *Group Steiner Tree* (GST) problem is a classical problem in combinatorial optimization and theoretical computer science. In the Edge-Weighted Group Steiner Tree (EW-GST) problem, we are given an undirected graph $G = (V, E)$ on n vertices with edge costs $c : E \rightarrow \mathbb{R}_{\geq 0}$, a source vertex s and a collection of subsets of vertices, called *groups*, $S_1, \ldots, S_k \subseteq V$. The goal is to find a minimum-cost tree $H \subseteq G$ that connects s to some vertex from each group S_i, for all $i = 1, 2, \ldots, k$. The Node-Weighted Group Steiner Tree (NW-GST) problem has the same setting, but the costs are associated with nodes. The goal is to find a minimum-cost node set $X \subseteq V$ such that $G[X]$ connects every group to the source. When G is a tree, both EW-GST and NW-GST admit a polynomial-time $O(\log n \log k)$ approximation algorithm due to the seminal result of (N. Garg, G. Konjevod, and R. Ravi, *A polylogarithmic approximation algorithm for the group steiner tree problem*, J. Algorithms 37(1):66–84, 2000, Preliminary version in SODA'98). The matching hardness of $\log^{2-\epsilon} n$ is known even for tree instances of EW-GST and NW-GST (E. Halperin and R. Krauthgamer, *Polylogarithmic inapproximability*, In Proceedings of the 35th Annual ACM Symposium on Theory of Computing, June 9-11, 2003, San Diego, CA, USA, pages 585–594, 2003). In general graphs, most of polynomial-time approximation algorithms for EW-GST reduce the problem to a tree instance using the metric-tree embedding, incurring a loss of $O(\log n)$ on the approximation factor (Y. Bartal, *Probabilistic approximations of metric spaces and its algorithmic applications*, In 37th Annual Symposium on Foundations of Computer Science, FOCS '96, Burlington, Vermont, USA, 14-16 October, 1996, pages 184–193, 1996), (J. Fakcharoenphol, S. Rao, and K. Talwar, *A tight bound on approximating arbitrary metrics by tree metrics*, J. Comput. Syst. Sci. 69(3):485–497, 2004, Preliminary version in STOC'03). This yields an approximation ratio of $O(\log^2 n \log k)$ for EW-GST. Using metric-tree embedding, this factor cannot be improved: The loss of $\Omega(\log n)$ is necessary on some input graphs (e.g., grids and expanders). There are alternative approaches that avoid metric-tree embedding, e.g., the algorithm of (C. Chekuri and M. Pál, *A recursive greedy algorithm for walks in directed graphs*, In 46th Annual IEEE Symposium on Foundations of Computer Science (FOCS 2005), 23-25 October 2005, Pittsburgh, PA, USA, Proceedings, pages 245–253, 2005), which gives a tight approximation ratio, but none of which achieves polylogarithmic approximation in polynomial-time. This state of the art shows a clear lack of understanding of GST in general graphs beyond the metric-tree embedding technique. For NW-GST (for which the metric-tree embedding does not apply), not even a polynomial-time polylogarithmic approximation algorithm is known. In this paper, we present $O(\log n \log k)$ approximation algorithms that run in time $n^{\tilde{O}(tw(G)^2)}$ for both NW-GST and EW-GST, where $tw(G)$ denotes the treewidth of graph G. The key to both results is a different type of "tree-embedding" that produces a tree of much bigger size, but *does not cause any loss on the approximation factor.* Our embedding is inspired by dynamic programming, a technique which is typically not applicable to Group Steiner problems.