Abstract

In the Minimum k-Union problem (MkU) we are given a set system with n sets and are asked to select k sets in order to minimize the size of their union. Despite being a very natural problem, it has received surprisingly little attention: the only known approximation algorithm is an $O(\sqrt{n})$-approximation due to [Chlamtáč et al APPROX '16]. This problem can also be viewed as the bipartite version of the Small Set Vertex Expansion problem (SSVE), which we call the Small Set Bipartite Vertex Expansion problem (SSBVE). SSVE, in which we are asked to find a set of k nodes to minimize their vertex expansion, has not been as well studied as its edge-based counterpart Small Set Expansion (SSE), but has recently received significant attention, e.g. [Louis-Makarychev APPROX '15]. However, due to the connection to Unique Games and hardness of approximation the focus has mostly been on sets of size $k = \Omega(n)$, while we focus on the case of general k, for which no polylogarithmic approximation is known. We improve the upper bound for this problem by giving an $n^{1/4+\varepsilon}$ approximation for SSBVE for any constant $\varepsilon > 0$. Our algorithm follows in the footsteps of Densest k-Subgraph (DkS) and related problems, by designing a tight algorithm for random models, and then extending it to give the same guarantee for arbitrary instances. Moreover, we show that this is tight under plausible complexity conjectures: it cannot be approximated better than $O(n^{1/4})$ assuming an extension of the so-called "Dense versus Random"" conjecture for DkS to hypergraphs. In addition to conjectured hardness via our reduction, we show that the same lower bound is also matched by an integrality gap for a super-constant number of rounds of the Sherali-Adams LP hierarchy, and an even worse integrality gap for the natural SDP relaxation. Finally, we note that there exists a simple bicriteria $\widetilde{O}(\sqrt{n})$ approximation for the more general SSVE problem (where no non-trivial approximations were known for general k).