Abstract

We study a wide spectrum of incidence problems involving points and curves or points and surfaces in $\mathbb{R}^3$. The current (and in fact the only viable) approach to such problems, pioneered by Guth and Katz, requires a variety of tools from algebraic geometry, most notably (i) the polynomial partitioning technique, and (ii) the study of algebraic surfaces that are ruled by lines or, in more recent studies by Guth and Zahl, by algebraic curves of some constant degree. By exploiting and refining these tools, we obtain new and improved bounds for numerous incidence problems in $\mathbb{R}^3$. In broad terms, we consider two kinds of problems, those involving points and constant-degree algebraic *curves*, and those involving points and constant-degree algebraic *surfaces*. In some variants we assume that the points lie on some fixed constant-degree algebraic variety, and in others we consider arbitrary sets of points in 3-space. The case of points and curves has been considered in several previous studies, starting with Guth and Katz's work on points and lines. Our results, which are based on a recent work of Guth and Zahl concerning surfaces that are doubly ruled by curves, provide a grand generalization of all previous results. We reconstruct the bound for points and lines, and improve, in certain significant ways, recent bounds involving points and circles (by Sharir Sheffer and Zahl) and points and arbitrary constant-degree algebraic curves (by Sharir, Sheffer and Solomon). While in these latter instances the bounds are not known (and are strongly suspected not) to be tight, our bounds are, in a certain sense, the best that can be obtained with this approach, given the current state of knowledge. In the case of points and surfaces, the incidence graph between them can contain large complete bipartite graphs, each involving points on some curve and surfaces containing this curve (unlike earlier studies, we do not rule out this possibility, which makes our approach more general). Our bounds estimate the total size of the *vertex sets* in such a complete bipartite graph decomposition of the incidence graph. In favorable cases, our bounds translate into actual incidence bounds. Overall, here too our results can be regarded as providing a "grand generalization" of most of the previous studies of (special instances of) this problem. As applications of our point-surface incidence bounds, we consider the problems of distinct and repeated distances determined by a set of n points in $\mathbb{R}^3$, two of the most celebrated open problems in combinatorial geometry. We obtain new and improved bounds for two special cases, one in which the points lie on some algebraic variety of constant degree, and one involving distances between pairs in $P_1 \times P_2$, where P_1 is contained in a variety and P_2 is arbitrary.