Abstract

Let p be a fixed prime. A triangle in $\mathbb{F}_p^n$ is an ordered triple (x, y, z) of points satisfying $x + y + z = 0$. Let $N = p^n = |\mathbb{F}_p^n|$. Green proved an arithmetic triangle removal lemma which says that for every $\epsilon > 0$ and prime p, there is a $\delta > 0$ such that if $X, Y, Z \subset \mathbb{F}_p^n$ and the number of triangles in $X \times Y \times Z$ is at most δN^2, then we can delete ϵN elements from X, Y, and Z and remove all triangles. Green posed the problem of improving the quantitative bounds on the arithmetic triangle removal lemma, and, in particular, asked whether a polynomial bound holds. Despite considerable attention, prior to this paper, the best known bound, due to the first author, showed that $1/\delta$ can be taken to be an exponential tower of twos of height logarithmic in $1/\epsilon$. We solve Green's problem, proving an essentially tight bound for Green's arithmetic triangle removal lemma in $\mathbb{F}_p^n$. We show that a polynomial bound holds, and further determine the best possible exponent. Namely, there is a computable number C_p such that we may take $\delta = (\epsilon/3)^{C_p}$, and we must have $\delta \le \epsilon^{C_p - o(1)}$. In particular, $C_2 = 1 + 1/(5/3 - \log_2 3) \approx 13.239$, and $C_3 = 1 + 1/c_3$ with $c_3 = 1 - \frac{\log b}{\log 3}$, $b = a^{-2/3} + a^{1/3} + a^{4/3}$, and $a = \frac{\sqrt{33}-1}{8}$, which gives $C_3 \approx 13.901$. The proof uses Kleinberg, Sawin, and Speyer's essentially sharp bound on multicolored sum-free sets, which builds on the recent breakthrough on the cap set problem by Croot-Lev-Pach, and the subsequent work by Ellenberg-Gijswijt, Blasiak-Church-Cohn-Grochow-Naslund-Sawin-Umans, and Alon.