Abstract

Graph coloring is arguably the most exhaustively studied problem in the area of approximate counting. It is conjectured that there is a fully polynomial-time (randomized) approximation scheme (FPTAS/FPRAS) for counting the number of proper colorings as long as $q \geq \Delta + 1$, where q is the number of colors and Δ is the maximum degree of the graph. The bound of $q = \Delta + 1$ is the uniqueness threshold for Gibbs measure on Δ-regular infinite trees. However, the conjecture remained open even for any fixed $\Delta \geq 3$ (The cases of $\Delta = 1, 2$ are trivial). In this paper, we design an FPTAS for counting the number of proper 4-colorings on graphs with maximum degree 3 and thus confirm the conjecture in the case of $\Delta = 3$. This is the first time to achieve this optimal bound of $q = \Delta + 1$. Previously, the best FPRAS requires $q > \frac{11}{6}\Delta$ and the best deterministic FPTAS requires $q > 2.581\Delta + 1$ for general graphs. In the case of $\Delta = 3$, the best previous result is an FPRAS for counting proper 5-colorings. We note that there is a barrier to go beyond $q = \Delta + 2$ for single-site Glauber dynamics based FPRAS and we overcome this by correlation decay approach. Moreover, we develop a number of new techniques for the correlation decay approach which can find applications in other approximate counting problems.