## A New Formula for $\pi$ Related to a Series of Ramanujan

*Problem* 11-003, *by* Jesús Guillera[1] (Ave. Cesaro Alierta, 31, esc. iz. 4°–A, Zaragoza, Spain).

*Conjecture.* Let

$$f(x)=\sum_{n=0}^{\infty}\frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3}\frac{1}{2^{6n}}\left(\frac{\sqrt{5}-1}{2}\right)^{8n}\left((42\sqrt{5}+30)(n+x)+(5\sqrt{5}-1)\right), \tag{1}$$

where $(a)_0=1$ and $(a)_n=a(a+1)\cdots(a+n-1)$ for $n>0$. On the basis of numerical calculations, our guess is that

$$f\left(\tfrac{1}{2}\right)=\frac{56+24\sqrt{5}}{15}\pi^2,$$

so there is a new identity satisfied by $\pi$,

$$\sum_{n=0}^{\infty}\frac{(\sqrt{5}-2)(51n+35)-3(n+1)}{\binom{2n}{n}^3(2n+1)^3}\left(\frac{\sqrt{5}-1}{2}\right)^{8n}=\frac{8\pi^2}{15}. \tag{2}$$

Prove this identity.

*Some Background.* In [6] Srinivasa Ramanujan used modular equations to prove three similar series involving $(1/2)_n^3/(1)_n^3$, namely

$$\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}\frac{1}{2^{2n}}(6n+1)=\frac{4}{\pi},\qquad \sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}\frac{1}{2^{6n}}(42n+5)=\frac{16}{\pi}, \tag{3}$$

and

$$\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^3}{(1)_n^3}\frac{1}{2^{6n}}\left(\frac{\sqrt{5}-1}{2}\right)^{8n}\left((42\sqrt{5}+30)n+(5\sqrt{5}-1)\right)=\frac{32}{\pi}. \tag{4}$$

A beautiful survey of this kind of series is given in [1], and two excellent books are [2] and [3].

Using the method of Wilf and Zeilberger in [5], we proved in [4] the identities

$$\sum_{n=0}^{\infty}\frac{1}{2^{2n}}\frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3}(6(n+x)+1)=8x\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n^2}{(x+1)_n^2} \tag{5}$$

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and

$$\sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{\left(x+\frac{1}{2}\right)_n^3}{(x+1)_n^3} (42(n+x)+5) = 32x \sum_{n=0}^{\infty} \frac{\left(x+\frac{1}{2}\right)_n^2}{(2x+1)_n^2}. \tag{6}$$

Substituting $x=\frac{1}{2}$ in (5) and (6), we obtained, respectively, the formulas

$$\sum_{n=0}^{\infty} \frac{1}{2^{2n}} \frac{(1)_n^3}{\left(\frac{3}{2}\right)_n^3} (6n+4) = \frac{\pi^2}{2}, \qquad \sum_{n=0}^{\infty} \frac{1}{2^{6n}} \frac{(1)_n^3}{\left(\frac{3}{2}\right)_n^3} (42n+26) = \frac{8\pi^2}{3}. \tag{7}$$

*Status.* This problem is open.

## REFERENCES


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