A Golden Example Solved

In Problem 06-003, Jonathan Borwein[1] (Dalhousie University, Halifax, NS, Canada) and Marc Chamberland[2] (Grinnell College, Grinnell, IA) asked for an evaluation of

$$\Phi := \sum_{k=0}^{\infty} \left\{ \frac{G^2}{(5k+1)^2} - \frac{G}{(5k+2)^2} - \frac{G^2}{(5k+3)^2} + \frac{G^5}{(5k+4)^2} + \frac{2\,G^5}{(5k+5)^2} \right\} g^{5k}, \tag{1}$$

where $g := (\sqrt{5}-1)/2$ and $G := (\sqrt{5}+1)/2 = g^{-1}$. The constant G is the *golden ratio* and is more often denoted by ϕ.

Solution by the proposers. The answer is $\Phi = \pi^2/50$. This was discovered empirically by Benoit Cloitre[3] using integer relation methods [1]. In the irrational base g, the constant Φ has digits independently computable since (1) gives an identity of BBP type. (See [1, Chapter 4]. Also, the site `mathworld.wolfram.com/BBP-TypeFormula.html` has a brief introduction, as does `www.answers.com/topic/bailey-borwein-plouffe-formula`.)

With computer algebra assistance, we worked backwards to needing to show

$$\Phi = \operatorname{Re} \operatorname{Li}_2 \left(2\cos(\theta)\, e^{\pi\, i\, \theta} \right)$$

for $\theta := 2\pi/5$, where $\operatorname{Li}_2(x) := \sum_{n=1}^{\infty} x^n/n^2$ is the *dilogarithm* [1, p. 210], [2].

Proposition. *For all real* θ

$$\sum_{n=1}^{\infty} \frac{\cos(n\theta)\,(2\,\cos(\theta))^n}{n^2} = \left(\frac{\pi}{2} - \theta\right)^2.$$

Proof. This is equation (5.17) in Lewin [2] and is relatively easy to obtain. □

Setting $\theta = 2\pi/5$ makes $2\cos(\theta) = (\sqrt{5}-1)/2 = g$. Moreover $2\cos(n\theta)$ takes the values 2, g, $-1/g$, $-1/g$, g modulo 5. With a little care, we obtain the golden mean BBP formula (1). □

Other Examples. Any other rational multiple of π has a like form—nicest when $2\cos(n\theta)$ is rational or quadratic. Thus, with $\theta := \pi/3, 2\pi/3$ we obtain, inter alia, an evaluation of $\zeta(2)$. With $\theta := 3\pi/8, 5\pi/12$ we obtain nice convergent identities.

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[2] Lewin, *Polylogarithms and Associated Functions*, Elsevier/North–Holland, New York, Amsterdam, 1981.