A Simple Proof of an Asymptotic Formula for a Sum of Cosecants

Solution of Problem 05-001 *by* Michael Renardy[1] (Virginia Tech).

We apply the trapezoidal rule to the integral

$$\int_{1/(2n)}^{1-1/(2n)} \left(\frac{1}{\sin(\pi x)} - \frac{1}{\pi x(1-x)} \right) dx.$$

This yields

$$\int_{1/(2n)}^{1-1/(2n)} \left(\csc(\pi x) - \frac{1}{\pi x(1-x)} \right) dx = \frac{1}{n} \sum_{k=1}^{n-1} \csc\left(\frac{\pi k}{n}\right) - \frac{n}{\pi} \sum_{k=1}^{n-1} \frac{1}{k(n-k)} + O(n^{-2}).$$

Next, we note that

$$\begin{aligned} \int_{1/(2n)}^{1-1/(2n)} \csc \pi x \, dx &= \frac{1}{\pi} \left[\ln\left(\tan \frac{\pi(2n-1)}{4n} \right) - \ln \tan \frac{\pi}{4n} \right] \\ &= -\frac{2}{\pi} \ln \tan \frac{\pi}{4n} = \frac{2}{\pi} \ln \frac{4n}{\pi} + O(n^{-2}), \end{aligned}$$

$$\int_{1/(2n)}^{1-1/(2n)} \frac{1}{\pi x(1-x)} \, dx = \frac{4}{\pi} \operatorname{Artanh}\left(1 - \frac{1}{n}\right) = \frac{2}{\pi}(\ln 2 + \ln n) + O(n^{-1}),$$

and

$$\sum_{k=1}^{n-1} \frac{1}{k(n-k)} = \frac{2}{n} \sum_{k=1}^{n-1} \frac{1}{k} = \frac{2}{n} \left(\ln n + \gamma + O(n^{-1}) \right).$$

The result follows easily by combining these formulae.

Also solved by Said Amghibech *and the proposer.*

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