Comment on an Asymptotic Formula for a Sum of Cosecants

In Problem 05-001, Hongwei Chen (Christopher Newport University) asked for a proof that

$$\sum_{k=1}^{n-1} \csc\left(\frac{k\pi}{n}\right) = \frac{2n}{\pi}\left(\ln n + \gamma - \ln\left(\frac{\pi}{2}\right)\right) + O(1), \qquad n \to \infty.$$

Michael Renardy has given an elementary proof.

Comment by V. E. Sandor Szabo[1] (Budapest University of Technology and Economics, Budapest, Hungary).

In refined form, this is given as an exercise in [1, Exercise 13, p. 460], and again in [2, Chapter 1, Exercise 23]. These references give the full asymptotic expansion of

$$\lambda_n := \frac{2}{n}\sum_{k=1}^{n-1} \csc\left(\frac{k\pi}{n}\right),$$

namely

$$\begin{aligned}\frac{\pi}{4}\,\lambda_n - \ln\frac{2n}{\pi} &\approx \gamma + \sum_{k=1}^{\infty} \frac{2^{2k-1}-1}{k(2k)!} B_{2k}^2 \left(-\frac{\pi^2}{n^2}\right)^k \\ &= \gamma - \frac{1}{72}\frac{\pi^2}{n^2} + \frac{7}{43200}\frac{\pi^4}{n^4} - \cdots, \qquad n \to \infty.\end{aligned}$$

Henrici credits this result to P. Waldvogel (private communication).

REFERENCES

[1] P. Henrici, *Applied and Computational Complex Analysis*, Vol. 2, John Wiley & Sons, New York, 1977.

[2] R. Wong, *Asymptotic Approximations of Integrals*, Classics in Applied Mathematics 34, SIAM, Philadelphia, 2001.

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