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Probabilistic Transportation Problem (PTP)

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Abstract

We study approximations of optimization problems with probabilistic constraints in which the original distribution of the underlying random vector is replaced with an empirical distribution obtained from a random sample. We show that such a sample approximation problem with risk level larger than the required risk level will yield a lower bound to the true optimal value with probability approaching one exponentially fast. This leads to an a priori estimate of the sample size required to have high confidence that the sample approximation will yield a lower bound. We then provide conditions under which solving a sample approximation problem with a risk level smaller than the required risk level will yield feasible solutions to the original problem with high probability. Once again, we obtain a priori estimates on the sample size required to obtain high confidence that the sample approximation problem will yield a feasible solution to the original problem. Finally, we present numerical illustrations of how these results can be used to obtain feasible solutions and optimality bounds for optimization problems with probabilistic constraints.

1. Introduction

A set of suppliers I and a set of customers D with $|D| = m$. The suppliers have limited capacity M_i for $i \in I$. There is a transportation cost c_{ij} for shipping a unit of product from supplier $i \in I$ to customer $j \in D$. The customer demands are random and are represented by a random vector d taking values in R^m. We assume we must choose the shipment quantities before the customer demands are known. We enforce the probabilistic constraint

$$P\left\{\sum_{i \in I} x_{ij} \geq d_j, j = 1,2,\ldots,m\right\} \geq 1-\epsilon \qquad (1)$$

Where $x_{ij} \geq 0$ is the amount shipped from supplier $i \in I$ to customer $j \in D$. The objective is to minimize distribution costs subject to (1), and the supply capacity constraints

$$\sum_{j \in D} x_{ij} \leq M_i \quad , \quad \forall i \in I \qquad (2)$$

2. Test

A test is conducted with 40 suppliers and 50 customers. The supply capacities and cost coefficients were randomly generated using normal and uniform distributions respectively. The demand is assumed to have a joint normal distribution. The mean vector and covariance matrix were randomly generated. We considered two cases for the covariance matrix: a low variance and a high variance case.

In the low variance case, the standard deviation of the one-dimensional marginal random demands is 10% of the mean on average. In the high variance case, the covariance matrix of the low variance case is multiplied by 25, yielding standard deviations of the one-dimensional marginal random demands being 50% of the mean on average. In both cases, we consider a single risk level $\epsilon = 0.05$.

We remark that for this particular choice of distribution, the feasible region defined by the probabilistic constraint is convex [23]. However, the dimension of the random vector d is m = 50, and so evaluating $P\{y \geq d\}$ for a single vector $y \in R^m$ is difficult. Hence, applying variations of standard convex programming techniques will not likely be efficient. However, generating random samples from the joint normal distribution is easy so that generating (non-convex) sample approximation problems can be accomplished.

Once a sample approximation is solved yielding solution x, we use a single very large sample ($N^0 = 250{,}000$), to estimate $P\{y^* \geq d\}$ where $y^* \in R^m$ is the vector given by $y_j = \sum_{i \in I} x_{ij}$ for $j \in D$.

Letting $d^1; : : : ; d^{N0}$ be the realizations of this large sample, we calculate $\sum_{i=1}^{N^0} \mathbb{I}\left(y \geq d^i\right)$ and use the normal approximation to the binomial distribution to construct an upper bound α on the true solution risk $P\{y^* \geq d\}$, which is valid with confidence 0.999. Henceforth for this experiment, if we say a solution is feasible at risk level ϵ, we mean $\alpha \leq \epsilon$, and so it is feasible at this risk level with confidence 0.999. We used such a large sample to get a good estimate of the true risk of the solutions generated, but we note that because this sample was so large, generating this sample and calculating $\sum_{i=1}^{N^0} \mathbb{I}(y \geq d^i)$ often took longer than solving the sample approximation itself.

3. Solution

We solved the sample approximation problem using a mixed-integer programming formulation, augmented with a class of strong valid inequalities. We refer the reader to [19, 18] for details of this formulation and the valid inequalities, as well as detailed computational results for solving the sample approximation problems. However, we mention that in contrast to the probabilistic set cover problem, solving the sample approximation problem with the largest sample size we consider (N = 10000) and largest α (0.05) takes a nontrivial amount of time, in some cases as long as 30 minutes. On the other hand, for N = 5000, the worst case was again α = 0.05 and usually took less than 4 minutes to solve.

4. Low Variance Case

Table 1 : Solution results for low variance PTP sample problems with ϵ = 0.05.

		Solution Risk				Feasible Solutions Cost				
α	N	Ave	Min	Max	σ	#	Ave	Min	Max	σ
0.000	900	0.048	0.036	0.068	0.011	7	2.0266	2.0198	2.0320	0.0045
	950	0.047	0.039	0.053	0.005	6	2.0244	2.0185	2.0291	0.0041
	1000	0.045	0.040	0.051	0.004	8	2.0253	2.0185	2.0300	0.0039
	1500	0.035	0.025	0.043	0.005	10	2.0336	2.0245	2.0398	0.0053
0.030	5000	0.049	0.045	0.050	0.002	6	2.0098	2.0073	2.0114	0.0013
	7500	0.045	0.041	0.047	0.002	10	2.0112	2.0094	2.0136	0.0015
	10000	0.042	0.041	0.044	0.001	10	2.0129	2.0112	2.0145	0.0016
0.033	5000	0.052	0.049	0.054	0.002	2	2.0080	2.0073	2.0088	0.0011
	7500	0.048	0.045	0.051	0.002	7	2.0092	2.0075	2.0107	0.0012
	10000	0.045	0.044	0.047	0.001	10	2.0103	2.0089	2.0118	0.0009
0.036	5000	0.055	0.053	0.057	0.002	0	***	***	***	***
	7500	0.052	0.049	0.054	0.002	2	2.0079	2.0077	2.0080	0.0002
	10000	0.049	0.047	0.051	0.001	8	2.0080	2.0066	2.0093	0.0008

We begin by presenting results for the instance in which the distribution of demand has relatively low variance. For generating feasible solutions, we tested α= 0 with various sample size N and report the results for the sample sizes, which yielded the best results. Once again, this means we use a relatively small sample size for the case α= 0, as compared to the cases with α> 0. We tested several values of α> 0 and varying sample size. In contrast to the PSC case, we found that taking α= ϵ or even α close to ϵ did not yield feasible solutions, even with a large sample size. Thus, we report results for several different values of α in the range 0.03 to 0.036. The reason we report results for this many different values of α is to illustrate that within this range, the results are not extremely sensitive to the choice of α (results for more values of α can be found in [18]).

Table 2 : Lower bounds for low variance PTP sample problems with $\alpha = \beta = 0.05$.

N	LB with confidence at least:				Gap with confidence at least:			
	0.999	0.989	0.945	0.828	0.999	0.989	0.945	0.828
1000	1.9755	1.9757	1.9775	1.9782	1.55%	1.54%	1.45%	1.42%
3000	1.9879	1.9892	1.9892	1.9910	0.93%	0.87%	0.87%	0.78%
5000	1.9940	1.9943	1.9948	1.9951	0.63%	0.62%	0.59%	0.57%
7500	1.9954	1.9956	1.9959	1.9963	0.56%	0.55%	0.54%	0.52%
10000	1.9974	1.9977	1.9980	1.9981	0.46%	0.45%	0.43%	0.42%

Table 1 gives the characteristics of the solutions generated for the different values of α and N. We observe that as in the case of the PSC, the average cost of the feasible solutions obtained using $\alpha > 0$ is always less than the minimum cost of the feasible solutions obtained with $\alpha = 0$. However, for this instance, the minimum cost solution obtained using $\alpha = 0$ is not so significantly worse than the minimum cost solutions using different values of $\alpha > 0$, being between 0.40% and 0.58% more costly. As in the case of the PSC, using $\alpha > 0$ and large N significantly reduced the variability of the risk and cost of the solutions generated.

We next investigated the quality of the lower bounds that can be obtained for PTP by solving sample approximation problems. As in the case of the PSC, we obtained lower bounds by generating and solving 10 sample approximation problems with $\alpha = \beta = 0.05$. By taking the lowest value of all the optimal values we obtain a lower bound valid with confidence 0.999, taking the second smallest yields a lower bound, which is valid with confidence 0.989, etc. The results for different values of N are given in Table 2. For reference, the percentage gap between these lower bounds and the best feasible solution found (with cost 2.0066) is also given. Using $N \geq 3000$ we obtain lower bounds that are valid with confidence 0.999 and are within one percent of the best feasible solution, indicating that for this low variance instance, the lower bounding scheme yields good evidence that the solutions we have found are good quality.

5. High Variance Case

Table 3 : Solution results for high variance PTP sample problems with $\beta = 0.05$.

α	N	Solution Risk				Feasible Solutions Cost				
		Ave	Min	Max	σ	#	Ave	Min	Max	σ
0.000	500	0.050	0.035	0.066	0.010	4	3.5068	3.4672	3.5488	0.0334
	750	0.050	0.041	0.058	0.006	6	3.4688	3.4403	3.4917	0.0191
	1000	0.045	0.041	0.052	0.004	9	3.4895	3.4569	3.5167	0.0234
	1500	0.030	0.022	0.035	0.005	10	3.5494	3.5205	3.6341	0.0368
0.030	5000	0.050	0.045	0.053	0.002	4	3.4014	3.3897	3.4144	0.0101
	7500	0.046	0.043	0.050	0.002	9	3.4069	3.3920	3.4235	0.0098
	10000	0.043	0.041	0.046	0.001	10	3.4139	3.4081	3.4181	0.0035
0.033	5000	0.053	0.048	0.057	0.003	1	3.4107	3.4107	3.4107	***
	7500	0.049	0.048	0.054	0.002	7	3.3928	3.3865	3.4029	0.0062
	10000	0.046	0.042	0.049	0.002	10	3.3982	3.3885	3.4139	0.0086
0.036	5000	0.057	0.049	0.060	0.003	1	3.3979	3.3979	3.3979	***
	7500	0.053	0.050	0.057	0.002	0	***	***	***	***
	10000	0.050	0.048	0.053	0.002	4	3.3927	3.3859	3.3986	0.0054

Table 3 gives the characteristics of the solutions generated for the high variance instance. In this case, the maximum cost of a feasible solution generated using any combination of $\alpha > 0$ and N was less than the minimum cost of any feasible solution generated using $\alpha = 0$. The minimum cost feasible solution generated with $\alpha = 0$ was between 0.87% and 1.6% more costly than the best feasible solution generated for the different combinations of $\alpha > 0$ and N. Thus, it appears that for the high variance instance, using $\alpha > 0$ in a sample approximation is more important for generating good feasible solutions than for the low variance instance.

Table 4 : Lower bounds for high variance PTP sample problems with $\alpha = \epsilon = 0{:}05$.

	LB with confidence at least:				Gap with confidence at least:			
N	0.999	0.989	0.945	0.828	0.999	0.989	0.945	0.828
1000	3.2089	3.2158	3.2178	3.2264	5.11%	4.91%	4.85%	4.59%
3000	3.2761	3.2775	3.2909	3.2912	3.12%	3.08%	2.69%	2.68%
5000	3.3060	3.3075	3.3077	3.3094	2.24%	2.19%	2.19%	2.14%
7500	3.3083	3.3159	3.3165	3.3169	2.17%	1.95%	1.93%	1.92%
10000	3.3200	3.3242	3.3284	3.3299	1.83%	1.70%	1.58%	1.53%

Table 4 gives the lower bounds for different confidence levels and sample sizes, as well as the gaps between these lower bounds and the best feasible solution found. In this case, solving 10 instances with sample size N = 1000 yields a lower bound that is not very tight, 5.11% from the best solution cost at confidence level 0.999. Increasing the sample size improves the lower bound, but even with N = 10000, the gap between the lower bound at confidence 0.999 and the best solution found is 1.83%. Thus, it appears that for the high variance instance, the sample approximation scheme exhibits considerably slower convergence, in terms of the lower bounds, the feasible solutions generated, or both.

6. Conclusion

We have studied a sample approximation scheme for probabilistically constrained optimization problems and demonstrated how this scheme can be used to generate optimality bounds and feasible solutions for very general optimization problems with probabilistic constraints. We have also conducted a preliminary computational study of this approach. This study demonstrates that using sample approximation problems that allow a choice of which sampled constraints to satisfy can yield good quality feasible solutions. In addition, the sample approximation scheme can be used to obtain lower bounds, which are valid with high confidence. We found that good lower bounds could be found in the case of finite (but possibly exponential) feasible region and distribution, and also in the case of continuous feasible region and distribution, provided the distribution has reasonably low variance. With continuous feasible region and distribution, if the distribution has high variance the lower bounds were relatively weak. Future work in this area will include conducting more extensive computational tests, and also extending the theory to allow generation of samples, which are not necessarily independent and identically distributed. For example, the use of variance reduction techniques such as Latin hypercube sampling or Quasi-Monte Carlo sampling may yield significantly faster convergence.

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