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The Transportation: More-for-less Criterion

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The Transportation: More-for-less Criterion

Abstract

The transportation more-for-less criterion is related to the classical transportation problem. For certain instances of this problem an increase in the amount of goods to be transported may lead to a decrease in the optimal total transportation cost. Even though the criteria has been known since the early days of linear programming, it has got very little attention in the literature, and it seems to be almost unknown to the majority of the LP-practitioners.

This paper presents necessary and sufficient conditions for a transportation cost matrix to be protected against the criteria. These conditions are rather restrictive; supporting the results reported from simulations that the criteria might occur quite frequently. We also consider some post optimal conditions for when the criteria may occur. A simple procedure for modifying an existing model to exploit the criteria is given and illustrated by examples.

Keywords:

Transportation Problem; Transportation Criteria; Linear Programming; Duality

1 Introduction

The increasing use of spatial analytical methods to identify optimal locations and efficient allocations has become a characteristic of operations in the private and public sectors. As companies face constant pressure for cost reduction and efficient operations, location and transportation strategies are required for efficient use of limited resources.

Various location and allocation objectives may be supported by the inclusion of constraints from the Natural Slack or Transportation Paradox frameworks. These constraints, when incorporated into existing models, provide opportunities for a More - for- Less solution which indicates improved system performance expressed either by expenditure savings, service improvements or both simultaneously. The philosophy behind the More-for-Less paradox is in tune with objectives of all classes of location models. Indeed, the More-for-Less paradox searches for additional benefit from a lower outlay of resources. Thus, the ability to exploit spatial relationships to extract extra efficiency should be a critical issue in location modeling.

The classical transportation problem is the name of a mathematical model, which has a special mathematical structure. The mathematical formulation of a large number of problems conforms (or can be made to conform) to this special structure. So the name is frequently used to refer to a particular form of mathematical model rather than the physical situation in which the problem most natural originates.

The standard problem description is as follows:

A commodity is to be transported from each of m sources to each of n destinations. The amounts available at each of the sources are a_i, i = 1, . . . ,m, and the demands at the destinations are b_j , j = 1, . . . , n respectively. The total sum of the available amounts at the sources is equal to the sum of the demands at the destinations. The cost of transporting one unit of the commodity from source i to destination j is c_{ij} .

The goal is to determine the amounts x_{ij} to be transported over all routes (i, j) such that the total transportation cost is minimized. The mathematical formulation of this standard version of the transportation problem is the following linear program, TP:

$$\textbf{minimize } \mathbf{Z} = \sum_{i=1}^{m} \sum_{j=1}^{n} \mathbf{x_{ij} c_{ij}}$$

$$\textbf{subject to} \quad \sum_{j=1}^{n} \mathbf{x_{ij}} = \mathbf{a_i} \qquad \textbf{for } \mathbf{i = 1,2,.....m}$$

$$\sum_{i=1}^{m} \mathbf{x_{ij}} = \mathbf{b_j} \qquad \textbf{for } \mathbf{j = 1,2,.....n}$$

$$\textbf{and} \qquad \mathbf{x_{ij} \geq 0} \qquad \forall\, (\mathbf{i, j})$$

An instance of TP is specified by an m × n cost matrix C = [c_{ij}], an n-dimensional demand vector b = [b_j] and an m-dimensional supply vector a = [a_i]. All the data are assumed to be nonnegative real numbers. We will use the notation z(C, a, b) to denote the optimal objective value of an instance of TP specified by C, a and b.

The so-called transportation criterion is the name of the following behaviour of the transportation problem: Certain instances have the property that it is possible to decrease the optimal objective value by increasing the supplies and demands. More precisely, let $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ be two other supply and demand vectors, such that $\hat{\mathbf{a}} \geq$ a and $\hat{\mathbf{b}} \geq$ b. Then the criteria occurs if and only if z(C, $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$) < z(C, a, b).

Example:1:

Let

	$\mathbf{D_1}$	$\mathbf{D_2}$	**Supply**
$\mathbf{O_1}$	**50**	**300**	**5**
$\mathbf{O_2}$	**320**	**60**	**10**
Demand	**7**	**8**	

By MODI method, the optimum solution is given by

x_{11}=5, x_{12}=0, x_{21}=2 and x_{22}=8 with min Z = 1370.

Let us denote the same values in the matrices as follows:

$$\mathbf{C} = \begin{bmatrix} \mathbf{50} & \mathbf{300} \\ \mathbf{320} & \mathbf{60} \end{bmatrix}, \ \mathbf{a} = \begin{bmatrix} \mathbf{5} & \mathbf{10} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} \mathbf{7} & \mathbf{8} \end{bmatrix} \ \textbf{and } \mathbf{X} = \begin{bmatrix} \mathbf{5} & \mathbf{0} \\ \mathbf{2} & \mathbf{8} \end{bmatrix}$$

with Z(C,a,b)=1370.

Now on increasing a_1 and b_2 by one unit, i.e. let $\hat{\mathbf{a}} = \begin{bmatrix} \mathbf{6} & \mathbf{10} \end{bmatrix}$ **and** $\hat{\mathbf{b}} = \begin{bmatrix} \mathbf{7} & \mathbf{8} \end{bmatrix}$

The optimal solution is then: $\mathbf{X} = \begin{bmatrix} \mathbf{6} & \mathbf{0} \\ \mathbf{1} & \mathbf{9} \end{bmatrix}$ with z(C, $\hat{\mathbf{a}}$, $\hat{\mathbf{b}}$) = 1160. So one more unit transported will reduce the optimal cost by 210.

Now the main focus is to identify where the value is to be increased and its alterations to the remaining basic cells.

2 Historical facts

It is not quite clear when and by whom this criterion was first discovered. Monge first formulated the transportation problem itself in 1781, which was solved by geometrical means and Hitchcock ([11]) in 1941, and was independently treated by Koopmans and Kantorovich. In 1951 Dantzig gave the standard LP- formulation TP in [7] and applied the simplex method to solve it. Very efficient algorithms and corresponding software have been developed for solving it.

The transportation criterion is, however, hardly mentioned at all where the transportation problem is treated. Apparently, several researchers have discovered the criteria independently from each other. But most papers on the subject refer to the papers by Charnes and Klingman [6] and Szwarc [14] as the initial papers. In [6] Charnes and Klingman name it the more-for-less criteria, and they write: The criteria was first observed in the early days of linear programming history (by whom no one knows) and has been a part of the folklore known to some (e.g. A.Charnes and W.W.Cooper), but unknown to the great majority of workers in the field of linear programming.

According to [4], the transportation criteria is known as Doigs criteria at the London School of Economics, named after Alison Doig who used it in exams etc. around 1959 (Doig did not publish any paper on it).

Since the transportation criteria seems not to be known to the majority of those who are working with the transportation problem, one may be tempted to believe that this phenomenon is only an academic curiosity, which will most probably not occur, in any practical situation. But that seems not to be true. Experiments done by Finke [9], with randomly generated instances of the transportation problem of size 100×100 and allowing additional shipments (post optimal) show that the transportation costs can be reduced considerably by exploiting the criteria properties. More precisely, the average cost reductions achieved are reported to be 18.6% with total additional shipments of 20.5%.

In a recent paper [8], Deineko & al. develop necessary and sufficient conditions for a cost matrix C to be protected against the transportation criteria. These conditions are rather restrictive, supporting the observations by Finke.

3 Non-occurrence of the criteria

In [8] Deineko & al. give an exact characterization of all cost matrices C that are protected against the transportation criteria. A protected cost matrix satisfies $z(C, a, b) \leq z(C, \hat{\mathbf{a}}, \hat{\mathbf{b}})$ for all supply vectors a and $\hat{\mathbf{a}}$ with $a \leq \hat{\mathbf{a}}$ and for all corresponding demand vectors b and $\hat{\mathbf{b}}$ with $b \leq \hat{\mathbf{b}}$. So regardless of the choice of the supply and demand vectors, the transportation criterion does not arise when the cost matrix C is protected.

Theorem: 1:

A m×n cost matrix $C = [c_{ij}]$ is protected against the transportation criteria if and only if, for all integers q, r, s, t with $1 \leq q, s \leq m$, $1 \leq r, t \leq n$, $q \neq s$, $r \neq t$, the inequality

$$c_{qr} \leq c_{qt} + c_{sr} \qquad \textbf{(1)}$$

is satisfied.

Proof:

Let C be protected against the transportation criteria.

To prove that (1) is satisfied.

If possible, suppose (1) is not true.

i.e. $c_{qr} > c_{qt} + c_{sr}$ for some q, r, s, t.

Then consider an instance where component q of the supply vector, $a_q = 1$, and where all the other components are zero, i.e. $a_i = 0$, $i = 1, . . . m$, $i \neq q$. Similarly, let component r of the demand vector, $b_r = 1$, and let all the other components be zero, i.e. $b_j = 0$, $j = 1, . . . , n$, $j \neq r$.

Then clearly $z(C, a, b) = c_{qr}$.

Now let $\hat{\mathbf{a}}$ be a new supply vector which is different from a only in component s such that $\hat{\mathbf{a}}_s = 1$, and similarly let $\hat{\mathbf{b}}$ be different from b only in component t such that $\hat{\mathbf{b}}_t = 1$. Then $a \leq \hat{\mathbf{a}}$ and $b \leq \hat{\mathbf{b}}$.

In this new instance one unit may be sent directly from source q to destination t, and another unit may be sent from source s to destination r. The total cost of this is $c_{qt}+c_{sr}$. Our assumption then leads to $z(C, a, b) > z(C, \hat{\mathbf{a}}, \hat{\mathbf{b}})$, i.e. the criteria has occurred, which is contradiction to the statement.

Hence, (1) is true when C is protected.

Note:

1. A quadruple (q, r, s, t) is considered to be Good quadruple if it is satisfied by $c_{qr} \leq c_{qt} + c_{sr}$
2. Theorem (1) can be stated as, an m×n cost matrix $C = (c_{ij})$ is protected against the transportation criteria, if and only if C does not contain a bad quadruple.

Example 2 : Consider the 4×5 cost matrix

$$\begin{bmatrix} \mathbf{4} & \mathbf{15} & \mathbf{6} & \mathbf{13} & \mathbf{14} \\ \mathbf{16} & \mathbf{9} & \mathbf{22} & \mathbf{13} & \mathbf{16} \\ \mathbf{8} & \mathbf{5} & \mathbf{11} & \mathbf{4} & \mathbf{5} \\ \mathbf{12} & \mathbf{4} & \mathbf{18} & \mathbf{9} & \mathbf{10} \end{bmatrix} \qquad \mathbf{(2)}$$

Here we see immediately that $c_{14} > c_{11} + c_{34}$, which means that (1) is violated for $q = 1, r = 4, s = 3, t = 1$. So C is not protected against the transportation criteria.

Hence in O(mn) time whether or not a $m \times n$ cost matrix C is protected against the transportation criteria. ([8])

4 Occurrence of the criteria

The dual problem corresponding to the linear program TP is the following linear program, DP:

$$\mathbf{maximize} \sum_{i=1}^{m} \mathbf{a_i u_i} + \sum_{j=1}^{n} \mathbf{b_j v_j}$$

$$\text{subject to } u_i + v_j \le c_{ij},\ i = 1, \ldots, m;\ j = 1, \ldots, n. \qquad \mathbf{(3)}$$

Here the dual variables u_i and v_j correspond to the m first and the n last equations of TP respectively. It is well known that the constraint equations of TP are linearly dependent and that the rank of the constraint matrix is m+n-1. So one equation (any) is redundant and may be omitted. Thus any optimal solution to DP will not be unique. In the following we will assume that the first constraint equation of TP is omitted, and that the corresponding dual variable, u_1, is set to be zero, i.e. $u_1 = 0$ throughout.

Any basic solution of TP has $m + n - 1$ basic variables. Let $X = [x_{ij}]$ be an optimal basic solution (also called an optimal transportation tableau) of TP and let B be the set of index pairs (i,j) of all basic variables x_{ij} in X. Then we know from elementary LP-theory that $u_i + v_j - c_{ij} = 0$ for all $(i,j) \in B$, and that $x_{ij} = 0$ for all $(i,j) \notin B$.

The optimal objective value may then be written as:

$$\mathbf{Z}(\mathbf{C},\mathbf{a},\mathbf{b}) = \sum_{\mathbf{i}=1}^{\mathbf{m}}\sum_{\mathbf{j}=1}^{\mathbf{n}} \mathbf{c}_{\mathbf{ij}}\mathbf{x}_{\mathbf{ij}}$$

$$= \sum_{\mathbf{i}=1}^{\mathbf{m}}\sum_{\mathbf{j}=1}^{\mathbf{n}} (\mathbf{c}_{\mathbf{ij}} - \mathbf{u}_{\mathbf{i}} - \mathbf{v}_{\mathbf{j}})\mathbf{x}_{\mathbf{ij}} + \sum_{\mathbf{i}=1}^{\mathbf{m}}\sum_{\mathbf{j}=1}^{\mathbf{n}} (\mathbf{u}_{\mathbf{i}} + \mathbf{v}_{\mathbf{j}})\mathbf{x}_{\mathbf{ij}}$$

$$= \sum_{\mathbf{i}=1}^{\mathbf{m}} \mathbf{u}_{\mathbf{i}}\mathbf{a}_{\mathbf{i}} + \sum_{\mathbf{j}=1}^{\mathbf{n}} \mathbf{v}_{\mathbf{j}}\mathbf{b}_{\mathbf{j}}$$

We will now look at some post optimal conditions, which are sufficient for the occurrence of the criteria. We consider only the case where an instance is improved by increasing a single supply a_i and a single demand b_j by the same amount (all the other data are unchanged). A procedure for improving an optimal transportation tableau when the conditions are satisfied will be illustrated.

Theorem: 2:

Assume that indexes p and q exist, $1 \le p \le m$; $1 \le\le q \le n$, such that

$$u_p + v_q < 0. \qquad \textbf{(4)}$$

Assume further that a positive number θ exists, such that when supply ap is replaced by $\hat{\mathbf{a}}_p = a_p + \theta$, and demand b_q is replaced by $\hat{\mathbf{b}}_q = b_q + \theta$, a basic feasible solution for the new instance can be found which is optimal and has the same set B of basic variables. Then the criteria will occur.

Proof :

Since the optimal solution for the new instance has the same set B of basic variables, the optimal dual solution is unchanged. So the new optimal objective value is:

$$\mathbf{Z}\left(\mathbf{C},\hat{\mathbf{a}},\hat{\mathbf{b}}\right) = \sum_{\mathbf{i}=1}^{\mathbf{m}} \mathbf{u}_{\mathbf{i}}\mathbf{a}_{\mathbf{i}} + \mathbf{u}_{\mathbf{p}}\theta + \sum_{\mathbf{j}=1}^{\mathbf{n}} \mathbf{v}_{\mathbf{j}}\mathbf{b}_{\mathbf{j}} + \mathbf{v}_{\mathbf{q}}\theta$$

$$= \mathbf{Z}(\mathbf{C},\mathbf{a},\mathbf{b}) + \theta(\mathbf{u}_{\mathbf{p}} + \mathbf{v}_{\mathbf{q}})$$

$$< \mathbf{Z}(\mathbf{C},\mathbf{a},\mathbf{b})$$

$$\textbf{Since } \theta > \mathbf{0} \textbf{ and } \mathbf{u}_{\mathbf{p}} + \mathbf{v}_{\mathbf{q}} < \mathbf{0} \Rightarrow \theta(\mathbf{u}_{\mathbf{p}} + \mathbf{v}_{\mathbf{q}}) < \mathbf{0}$$

Therefore, $z(C, \hat{\mathbf{a}}, \hat{\mathbf{b}}) < z(C, a, b)$.

Existence of positive θ:

A 4×5 instance of TP is given by the cost matrix (2) in Example 2 and the following supply and demand vectors:

a = [7 18 6 15] and b = [4 11 12 8 11]

The optimal transportation tableau for this instance is:

	$\mathbf{v_1 = 0}$	$\mathbf{v_2 = -6}$	$\mathbf{v_3 = 6}$	$\mathbf{v_4 = -2}$	$\mathbf{v_5 = 0}$
$\mathbf{u_1 = 0}$			**7**		
$\mathbf{u_2 = 15}$	**4**	**6**		**8**	
$\mathbf{u_3 = 5}$			**5**		**1**
$\mathbf{u_4 = 10}$		**5**			**10**

- Here the optimal dual values are written above and on the left of the tableau.
- The total optimal cost of this solution is 444.
- We observe that the set of index pairs for the optimal basic variables is:

  B = {(1, 3), (2, 1), (2, 2), (2, 4), (3, 3), (3, 5), (4, 2), (4, 5)}.
- We also observe that $u_1 + v_4 = -2 < 0$.

So let us see if it is possible to increase $a_1 = 7$ and $b_4 = 8$ by a number $\theta > 0$ such that the present optimal basic feasible solution can be modified to become optimal for the new instance with the same set of basic variables:

		$\mathbf{7+\theta}$			$\mathbf{a_1 = 7+\theta}$
4	$\mathbf{6-\theta}$		$\mathbf{8+\theta}$		$\mathbf{a_2 = 18}$
		$\mathbf{5-\theta}$		$\mathbf{1+\theta}$	$\mathbf{a_3 = 6}$
	$\mathbf{5+\theta}$			$\mathbf{10-\theta}$	$\mathbf{a_4 = 15}$
$\mathbf{b_1 = 4}$	$\mathbf{b_2 = 11}$	$\mathbf{b_3 = 12}$	$\mathbf{b_4 = 8+\theta}$	$\mathbf{b_5 = 11}$	

Here the supplies and the demands are written to the right and below the tableau. From this tableau we observe that θ may be selected as any number $0 < \theta \leq 5$.

If $\theta = 4$ is chosen, the new optimal transportation tableau is:

		11			$\mathbf{a_1 = 11}$
4	**2**		**12**		$\mathbf{a_2 = 18}$
		1		**5**	$\mathbf{a_3 = 6}$
	9			**6**	$\mathbf{a_4 = 15}$
$\mathbf{b_1 = 4}$	$\mathbf{b_2 = 11}$	$\mathbf{b_3 = 12}$	$\mathbf{b_4 = 12}$	$\mathbf{b_5 = 11}$	

The cost of this solution is $z(C, \hat{\mathbf{a}}, \hat{\mathbf{b}}) = z(C,a,b) + \theta(u_p+v_q) = 444 + 4(-2) = 436$. So shipping 4 units more will reduce the total transportation cost by 8 units.

Note that if $\theta = 5$ is chosen (the maximum value for θ), the new optimal transportation tableau will be degenerate (one of the basic variables becomes zero).

We observe that in order to determine the upper bound for θ, a subset $S \subseteq B$ of index pairs has been selected ($S = B\backslash\{(2, 1)\}$ in our example). Now suppose we link the elements of S to form a directed path DS:

DS = {(1, 3), (3, 3), (3, 5), (4, 5), (4, 2), (2, 2), (2, 4)}.

- This ordered set defines a directed path, which starts at the basic element (1, 3) and ends at (2, 4).
- It is alternating in the sense that θ is added to the tableau elements corresponding to the odd numbered elements of DS and is subtracted from those corresponding to the even numbered elements of DS.
- DS consists of an even number of perpendicular links.
- DS will consist of index pairs for an odd number of (perpendicular) basic elements of the tableau.
- The change in the problem occurs when there exist the path.
- For more than one $u_i+v_j < 0$, maximum importance is to select the combination where i^{th} row and j^{th} column have only one basic cell. (Example 5 for more than one)

In general, if indexes p and q exist such that (4) is satisfied, try to determine an upper bound for θ by constructing a directed alternating path DS, starting at a basic element (p,-) in row p of the optimal transportation tableau and ending at a basic element (-, q) in column q (in example p = 1 and q = 4).

Let DS_o and DS_e denote the odd and even numbered elements of DS respectively, such that $DS = DS_o \cup DS_e$. The elements of the cost matrix C corresponding to the index pairs in DS are related by the following lemma.

Lemma 1:

$$\mathbf{CDS} = \sum_{(i,j)\in DS_0} \mathbf{c_{ij}} - \sum_{(i,j)\in DS_e} \mathbf{c_{ij}} = \mathbf{u_p} + \mathbf{v_q}$$

Proof:

Since the elements of DS are (perpendicular) alternating, we have that

$$\mathbf{CDS} = \mathbf{c_{pj1}} - \mathbf{c_{i_1 j_1}} + \mathbf{c_{i_1 j_2}} - \mathbf{c_{i_2 j_2}} + \ldots\ldots - \mathbf{c_{i_t j_t}} + \mathbf{c_{i_t q}}$$

Since $c_{ij} = u_i + v_j$ for all $(i,j) \in B$ and $S \subseteq B$, we have

$$\begin{aligned} \text{CDS} &= (u_p+v_{j1}) - (u_{i1}+v_{j1}) + (u_{i1}+v_{j2}) - \ldots\ldots\ldots - (u_{it}+v_{jt}) + (u_{it}+v_q) \\ &= u_p + v_q. \end{aligned}$$

We add θ to the tableau elements corresponding to DS_o and subtract θ from the tableau elements corresponding to DS_e. The upper bound for θ is limited by the smallest basic element of the optimal transportation tableau from which θ is subtracted. So we have the following result:

Corollary 1:

A positive θ exists if and only if $x_{ij} > 0$, □$(i, j) \in DS_e$.

This corollary tells us that if the optimal solution of TP is nondegenerate, and there are components of the optimal dual solution satisfying (4), the criteria will occur. In case of degeneracy the criteria will still occur if the index pairs of DS_e do not include any degenerate elements of the optimal tableau (Example 5).

Repeated use of the process is of course possible if more than one pair of optimal dual values satisfy (4) (the dual solution is unchanged). However, if the maximal value of θ is selected, the new optimal tableau will be degenerate, and this may reduce the possibility of repeated success (as Example 5 also shows).

Example 3:

There are three origins (plants) and four destinations (distribution centers). The amounts (number of units of the product) available at each of the origins, the demands at each of the destinations and the transportation cost from each origin i to each destination j are given respectively by:

$$a = [\ 5000\ 6000\ 2500\ ],\ b = [\ 6000\ 4000\ 2000\ 1500\ ] \text{ and } \mathbf{C} = \begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{7} & \mathbf{6} \\ \mathbf{7} & \mathbf{5} & \mathbf{2} & \mathbf{3} \\ \mathbf{2} & \mathbf{5} & \mathbf{4} & \mathbf{5} \end{bmatrix}$$

It is easy to verify that this cost matrix is not protected against the transportation criteria. The optimal transportation tableau for this instance is:

	$\mathbf{v_1 = 3}$	$\mathbf{v_2 = 2}$	$\mathbf{v_3 = -1}$	$\mathbf{v_4 = 0}$
$\mathbf{u_1 = 0}$	**3500**	**1500**		
$\mathbf{u_2 = 3}$		**2500**	**2000**	**1500**
$\mathbf{u_3 = -1}$	**2500**			

The optimal total cost is 39500.

The assumption (4) is satisfied by p = 3 and q = 3 since $u_3 + v_3 = -2 < 0$.

A directed path DS starting in (3, 1) and ending in (2, 3) can be constructed. Since the optimal solution is non-degenerate, we know from Corollary 1 that a positive number θ can be found, which when added to a_3 and b_3 will give a new optimal tableau with a lower cost.

To determine an upper bound for θ, we consider the following transportation tableau:

$\mathbf{3500-\theta}$	$\mathbf{1500+\theta}$			$\mathbf{a_1 = 5000}$
	$\mathbf{2500-\theta}$	$\mathbf{2000+\theta}$	**1500**	$\mathbf{a_2 = 6000}$
$\mathbf{2500+\theta}$				$\mathbf{a_3 = 2500+\theta}$
$\mathbf{b_1 = 6000}$	$\mathbf{b_2 = 4000}$	$\mathbf{b_3 = 2000+\theta}$	$\mathbf{b_4 = 1500}$	

We see that any value of θ such that $0 < \theta \leq 2500$ will give a new optimal solution with a lower cost. If $\theta = 2500$ is selected, the total cost is reduced by 5000 (to 34500). Repeated use of the process is then not possible (due to degeneracy, no positive θ can be found for the other dual combinations satisfying (4)).

Example 4:

Three power plants supply the needs for electricity of four cities. The number of kilowatt-hours (in millions) each power plant can supply, and the (peak) power demands at the four cities are respectively:

a = [35 50 40] and b = [45 20 30 30].

The cost of sending one million kwh from plant i to city j is given by the following matrix:

$$\mathbf{C} = \begin{bmatrix} \mathbf{8} & \mathbf{6} & \mathbf{10} & \mathbf{9} \\ \mathbf{9} & \mathbf{12} & \mathbf{13} & \mathbf{7} \\ \mathbf{14} & \mathbf{9} & \mathbf{16} & \mathbf{5} \end{bmatrix}$$

This matrix is not protected against the transportation criteria. But the optimal solution of this instance is:

	$v_1 = 6$	$v_2 = 6$	$v_3 = 10$	$v_4 = 2$
$u_1 = 0$		10	25	
$u_2 = 3$	45		5	
$u_3 = 3$		10		30

Here there are not any indexes p and q such that (4) is satisfied. So the criterion does not occur.

However, from Theorem 1 and Lemma 1 we see that if the supplies and demands were such that the optimal set B of index pairs included, the index pairs (1,3),(3,3) and (3,4), or the index pairs (1,1),(3,1) and (3,4), the criteria may occur. To confirm this, suppose we have another instance where the cost matrix C is the same, but the supplies and the demands are:

a = [45 20 60] and b = [35 20 40 30].

An optimal transportation tableau (not unique) for this instance is:

	$v_1 = 8$	$v_2 = 3$	$v_3 = 10$	$v_4 = -1$
$u_1 = 0$	15		30	
$u_2 = 1$	20			
$u_3 = 6$		20	10	30

The total cost is 1090.

Since $u_1+v_4 = -1$, the total cost will be reduced by θ if supply a_1 and demand b_4 are both increased by θ, where $0 < \theta \leq 10$.

Another optimal transportation tableau for this instance is:

	$v_1 = 8$	$v_2 = 3$	$v_3 = 10$	$v_4 = -1$
$u_1 = 0$	5		40	
$u_2 = 1$	20			
$u_3 = 6$	10	20		30

Again we see that the total optimal cost will decrease if we increase the same supply and demand as we did in the previous tableau.

Example 5:

Let a = [10 6 15 4], b = [5 14 10 6] and $C = \begin{bmatrix} 5 & 8 & 7 & 6 \\ 6 & 10 & 5 & 5 \\ 7 & 15 & 3 & 16 \\ 15 & 21 & 8 & 18 \end{bmatrix}$

The optimal TP-tableau for this instance is:

	$v_1 = 5$	$v_2 = 8$	$v_3 = -4$	$v_4 = 3$
$u_1 = 0$		10		
$u_2 = 2$		0		6
$u_3 = 7$	5	4	6	
$u_4 = 9$			4	

The total cost of this solution is 255.

Since $u_1 + v_3 = -4$ and $u_2 + v_3 = -2$, we have two possible starting points for improvements. If we start with the first alternative, we increase a_1 and b_3 by $\theta = 4$ and get the following tableau:

	14		
	0		6
5	0	10	
		4	

The total cost of this solution is 255 - 4 * 4 = 239. This solution cannot be further improved (trying to increase a_2 and b_3 yields $\theta = 0$).

But if we start to improve the optimal TP tableau by first increasing a_2 and b_3 (again by $\theta = 4$), we get the following tableau:

	10		
	4		6
5	0	10	
		4	

The total cost of this solution is 255 - 2 * 4 = 247. This solution cannot be improved further (trying to increase a_1 and b_3 yields $\theta = 0$).

6 Conclusions:

We have considered the classical transportation problem and studied the occurrence of the so-called transportation criteria (also called the more-for-less criteria). Even if the first discovery of the criteria is a bit unclear, it is evident that it has been known since the early days of LP. It has, however, got very little attention in the literature. The main reason for this may be that it is considered as a rather odd phenomenon, which hardly occurs, in any practical situation.

The simulation research reported by Finke in [9] indicates, however, that the criteria may occur quite frequently. The rather restrictive conditions for a cost matrix to be protected against the criteria (see Theorem 1) point in the same direction.

We therefore urge that the transportation criteria should be given much more attention. In addition we hope that a lot of the existing excellent software for TP will be extended to include at least a preprocessing routine for deciding whether the cost matrix is protected or not against the criteria. If the cost matrix is not protected, and there are optimal dual variables satisfying (4), an option allowing post processing of the optimal solution should be available. The cost of these additional computations is modest and may provide valuable new insight in the problem from which the data for the actual TP-instance originates.

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